Abstract

There has been a growing research interest in the areas of construction and application of multiwavelets over the past few years. In a previous paper, we introduced a class of symmetric-antisymmetric orthonormal multiwavelets which were constructed directly from orthonormal scalar wavelets. These multiwavelets were shown to perform better than existing orthonormal multiwavelets and scalar wavelets in terms of image compression performance and computational complexity. However, their performance still lags behind some popular biorthogonal scalar wavelets such as Daubechies' D(9/7) and Villasenor's V(10/18). This paper aims to address this shortcoming by extending our earlier work to the biorthogonal setting. Two methods of construction are introduced; thus resulting in previously unpublished symmetric-antisymmetric biorthogonal multiwavelet filters. Extensive simulations showed that these multiwavelet filters can give an improvement of up to 0.7 dB over D(9/7) and V(10/18), and yet require only comparable but often lower computational cost. More importantly, better preservation of textures and edges of the reconstructed images was also observed.

1 Introduction

The study of multiwavelets as an extension from scalar wavelets has received considerable attention from the wavelets research communities both in theory [4, 8] as well as in applications such as signal compression and denoising [16, 18, 22]. It was shown in [1, 4] that symmetry, orthogonality, compact support and approximation order $k > 1$ can be simultaneously achieved for multiwavelets although this is not possible for scalar wavelets. In our preceding paper [13], we introduced a class of symmetric-antisymmetric orthonormal multiwavelet filters (SAOMFs) which can be constructed directly in an easy manner from orthonormal scalar wavelets. Such a direct construction is made possible due to the concept of an *"equivalent system of scalar filters"* and the idea of *"good multifilter properties"* (GMPs), which we introduced earlier in [18, 19]. The first concept formulates a relationship in which any multiwavelet filter with multiplicity r can be sufficiently represented in terms of identical input-output relationships by an equivalent set of r scalar wavelet filters. The idea of GMPs provides a set of design criteria that can be imposed on the equivalent scalar filters, which in turn can be translated directly as eigenvector properties of the designed multiwavelet filters. The class of SAOMFs that possess GMPs have been shown to be useful for image compression. In addition, such a direct construction is attractive as the derived multiwavelet filters can possess linear phase which is lacking in the orthonormal scalar wavelets. By integrating the designed SAOMFs into a general framework for multiwavelet decomposition and reconstruction, we could achieve improved image compression performance with lower computational requirements.

In this paper, we will extend the study of symmetric-antisymmetric multiwavelet filters that possess GMPs to the biorthogonal case. Unlike orthonormal scalar wavelet filters, it is already possible to design biorthogonal scalar wavelets that have linear phase property. Nevertheless, the motivation of our extension work is twofold. First, we show that the proposed biorthogonal multiwavelet filters (BMFs) can provide a new decomposition and reconstruction framework for the application of even-length biorthogonal scalar wavelets. BMFs derived from existing even-length scalar wavelets can in fact produce improvement in image compression over the direct application of scalar wavelets. Second, the availability of matrix coefficients, rather than scalar coefficients, provides us with more design parameters (degrees of freedom) for designing multiwavelet filters than for designing scalar wavelet filters of the same filter lengths. For example, the length-4 scalar orthonormal wavelet D4 has a maximum vanishing moment of 2, while that for length-4 orthonormal multiwavelet of multiplicity 2 is 3 [1]. In this paper, we take advantage of these extra degrees of freedom to construct better multiwavelet filters that possess GMPs.

The rest of the paper is organised as follows. In Section 2, we begin with some relevant basic theory of BMFs with multiplicity r. In Section 3, we review the definition of GMPs in the context for constructing BMFs. In Section 4, we propose two methods for the construction of symmetric-antisymmetric BMFs (SABMFs) possessing GMPs. The first method shows that any even-length, linear-phase biorthogonal scalar wavelet filter can generate a SABMF. This also means that we can alternatively implement even-length biorthogonal scalar wavelet filters using the proposed multiwavelet framework. The second method shows that a pair of non-symmetric scalar sequences satisfying some given conditions can also generate a SABMF. Examples of SABMFs of different lengths constructed using both methods are given. In Section 5, we will describe a general framework for the application of multiwavelet filters, and address the problem of multiwavelet initialization or pre-filtering. An extensive performance analysis of the proposed families of SABMFs is carried out in Section 6. Comparisons of objective and subjective image compression performances as well as the computational complexity of various multiwavelet filters and scalar filters are made. Finally some suggestions for future research and the conclusions are drawn in Section 7.

Notation.

Bold-faced characters are used to denote vectors and matrices. The matrices $\boldsymbol{P}^T$ and $\boldsymbol{P}^{-1}$ denote respectively the transpose and the inverse of the matrix $\boldsymbol{P}$. In addition, $\boldsymbol{P}^\sharp$ denotes the similarity transformation of $\boldsymbol{P}$ using a transition matrix $\boldsymbol{U}$; that is, $\boldsymbol{P}^\sharp = \boldsymbol{U}\boldsymbol{P}\boldsymbol{U}^{-1}$. Symbols $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and zero matrices respectively. For a given function f, $\widehat{f}$ will denote its Fourier transform. For brevity, we will express the eigenvector of an operator $\boldsymbol{A}$ corresponding to an eigenvalue λ as the λ-*eigenvector* of $\boldsymbol{A}$. Throughout the paper, j will denote $\sqrt{-1}$.

2 Preliminaries of biorthogonal multiwavelet theory

We present here only a subset of the basic theory for biorthogonal multiwavelets necessary for our exposition. For a more complete and rigorous presentation, interested readers can refer to [3, 8].

A biorthogonal multiwavelet system (BMWS) consists of two multiscaling function vectors $\boldsymbol{\phi} := [\phi_1, \ldots, \phi_r]^T$ and $\widetilde{\boldsymbol{\phi}} = [\widetilde{\phi}_1, \ldots, \widetilde{\phi}_r]^T$ where $r > 1$ is an integer. The component functions here are referred to as the multiscaling functions. In this paper, we will restrict ourself to consider only multiscaling functions with compact support. The multiscaling functions generate a multiresolution analysis pair of $\{V_\ell\}_{\ell \in \mathbb{Z}}$ and $\{\widetilde{V}_\ell\}_{\ell \in \mathbb{Z}}$ of $L^2(\mathbb{R})$. An important concept is that $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ satisfy the refinement equations:

$$\boldsymbol{\phi}(x) = \sum_k \boldsymbol{H}_k \boldsymbol{\phi}(2x-k), \tag{1}$$

$$\widetilde{\boldsymbol{\phi}}(x) = \sum_k \widetilde{\boldsymbol{H}}_k \widetilde{\boldsymbol{\phi}}(2x-k) \tag{2}$$

for some finite length real-valued matrix sequences $\{\boldsymbol{H}_k\}_{k \in \mathbb{Z}}$ and $\{\widetilde{\boldsymbol{H}}_k\}_{k \in \mathbb{Z}}$. Associated with these $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ are biorthogonal multiwavelet vectors $\boldsymbol{\psi} = [\psi_1, \ldots, \psi_r]^T$ and $\widetilde{\boldsymbol{\psi}} = [\widetilde{\psi}_1, \ldots, \widetilde{\psi}_r]^T$ which are related to the multiscaling function vectors via the following equations:

$$\boldsymbol{\psi}(x) = \sum_k \boldsymbol{G}_k \boldsymbol{\phi}(2x-k), \tag{3}$$

$$\widetilde{\boldsymbol{\psi}}(x) = \sum_k \widetilde{\boldsymbol{G}}_k \widetilde{\boldsymbol{\phi}}(2x-k) \tag{4}$$

where $\{\boldsymbol{G}_k\}_{k \in \mathbb{Z}}$ and $\{\widetilde{\boldsymbol{G}}_k\}_{k \in \mathbb{Z}}$ are some finite length real-valued matrix sequences. The component functions of $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$ are referred to as multiwavelet functions.

For any BMWS, the sequences $\{\boldsymbol{H}_k\}_{k \in \mathbb{Z}}$ and $\{\widetilde{\boldsymbol{H}}_k\}_{k \in \mathbb{Z}}$ constitute the matrix lowpass filters whereas $\{\boldsymbol{G}_k\}_{k \in \mathbb{Z}}$ and $\{\widetilde{\boldsymbol{G}}_k\}_{k \in \mathbb{Z}}$ constitute the corresponding matrix highpass filters. Biorthogonality of the multiscaling and multiwavelet functions implies the following perfect reconstruction (PR) conditions on these matrix lowpass and highpass filters:

$$\sum_{k\in\mathbb{Z}} \boldsymbol{H}_k \widetilde{\boldsymbol{H}}_{k+2i}^T = 2\delta_i \boldsymbol{I}, \tag{5}$$

$$\sum_{k\in\mathbb{Z}} \boldsymbol{H}_k \widetilde{\boldsymbol{G}}_{k+2i}^T = \boldsymbol{0}, \tag{6}$$

$$\sum_{k\in\mathbb{Z}} \boldsymbol{G}_k \widetilde{\boldsymbol{G}}_{k+2i}^T = 2\delta_i \boldsymbol{I}, \tag{7}$$

where $i \in \mathbb{Z}$ and $\delta_i = 1$ if $i = 0$ and 0 otherwise. Specifically, the sequences $\{\boldsymbol{H}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\boldsymbol{H}}_k\}_{k\in\mathbb{Z}}$ which satisfy (5) constitute a matrix conjugate quadrature filter (CQF). We also define $\boldsymbol{H}(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{H}_k e^{-jk\omega}$ and $\widetilde{\boldsymbol{H}}(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{H}}_k e^{-jk\omega}$ which are the matrix lowpass frequency responses (or refinement masks) associated with the multiscaling functions. Each refinement mask is a $r \times r$ matrix with trigonometric polynomial entries. Similarly, we denote the matrix highpass frequency responses (or wavelet masks) as $\boldsymbol{G}(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{G}_k e^{-jk\omega}$ and $\widetilde{\boldsymbol{G}}(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{G}}_k e^{-jk\omega}$ respectively.

In this paper, our methods for constructing SABMFs involves mainly the construction of the matrix lowpass and highpass filters satisfying (5)–(7) and some other conditions. Filters constructed as such do not necessarily lead to multiscaling and multiwavelet function vectors of a BMWS. This only happens if the transition operators defined on $\boldsymbol{H}(\omega)$ and $\widetilde{\boldsymbol{H}}(\omega)$ satisfy Condition E [11].

In signal processing applications, an important property of the BMF is the approximation order of the associated multiscaling function vector. From [7], $\boldsymbol{\phi}$ provides approximation order m[1] if and only if its refinement mask $\boldsymbol{H}(\omega)$ satisfies the following conditions: there are vectors $\boldsymbol{y}_k \in \mathbb{R}^r$; $\boldsymbol{y}_0 \neq \boldsymbol{0}$, $k = 0, \ldots, m-1$, such that for $n = 0, \ldots, m-1$,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^T (2j)^{k-n} (D^{n-k}\boldsymbol{H})(0) = 2^{-n} (\boldsymbol{y}_n)^T, \\ \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^T (2j)^{k-n} (D^{n-k}\boldsymbol{H})(\pi) = \boldsymbol{0}^T, \end{aligned} \tag{8}$$

with the operator $D := \frac{d}{d\omega}$. A similar result applies to $\widetilde{\boldsymbol{\phi}}$ and its refinement mask $\widetilde{\boldsymbol{H}}(\omega)$. We say that a BMF has approximation order (m_1, m_2) if the refinement masks $\boldsymbol{H}(\omega)$ and $\widetilde{\boldsymbol{H}}(\omega)$ satisfy the above conditions for $m = m_1$ and $m = m_2$ respectively.

[1] A function vector $\boldsymbol{\phi}$ has an approximation order of m when all polynomials of degree from 0 to $m-1$ can be exactly reproduced by a linear combination of integer translates of ϕ_k, $k = 1, \ldots, r$.

3 Design Criteria for Biorthogonal Multiwavelet Filters

For simplicity of exposition, we will focus on designing BMFs with multiplicity $r = 2$ in this paper. Extensions to a higher multiplicity can also be carried out. In [18], we proposed a new design criterion called *"good multifilter properties"* (GMPs) as a useful tool for analyzing and constructing orthogonal multiwavelets. In this section, we will extend the concept of GMPs to the biorthogonal setting with the main aim of constructing some BMFs which can perform better than both popular orthogonal and biorthogonal scalar wavelet filters in image compression applications.

Now, for any BMF, we define a set of associated *equivalent scalar filters*, $\mathcal{H}_\nu(\omega)$, $\nu = 1, 2$, with the corresponding frequency responses [18]

$$\mathcal{H}_\nu(\omega) := \frac{1}{2}\sum_k \left(h_{\nu,1}(k)e^{-2jk\omega} + h_{\nu,2}(k)e^{-j(2k+1)\omega}\right), \quad \nu = 1, 2, \tag{9}$$

where $\boldsymbol{H}_k = (h_{\nu,n}(k))_{\nu,n=1}^2$. The definitions of $\mathcal{G}_\nu(\omega)$, $\widetilde{\mathcal{H}}_\nu(\omega)$ and $\widetilde{\mathcal{G}}_\nu(\omega)$ are similar to $\mathcal{H}_\nu(\omega)$. The masks $\boldsymbol{H}(\omega)$, $\boldsymbol{G}(\omega)$, $\widetilde{\boldsymbol{H}}(\omega)$ and $\widetilde{\boldsymbol{G}}(\omega)$ are the polyphase matrices of scalar filters $\mathcal{H}_\nu(\omega)$, $\mathcal{G}_\nu(\omega)$, $\widetilde{\mathcal{H}}_\nu(\omega)$ and $\widetilde{\mathcal{G}}_\nu(\omega)$, $\nu = 1, 2$, respectively. Since we usually need to further decompose the outputs that are filtered with $\mathcal{H}_\nu(\omega)$, we require $\mathcal{H}_\nu(\omega)$, $\nu = 1, 2$, to possess lowpass properties. On the contrary, since the outputs that are filtered with $\mathcal{G}_\nu(\omega)$ are kept for processing (e.g. quantization), we expect $\mathcal{G}_\nu(\omega)$ to exhibit bandpass properties so that only a small number of the outputs have significant energy ([13], [14], [18]). By using an invertible transition matrix $\boldsymbol{U}$, we can transform [9, 18] the refinement masks $\boldsymbol{H}(\omega)$, $\widetilde{\boldsymbol{H}}(\omega)$ and the wavelet masks $\boldsymbol{G}(\omega)$, $\widetilde{\boldsymbol{G}}(\omega)$ respectively to

$$\begin{aligned}
\boldsymbol{H}^\sharp(\omega) &= \boldsymbol{U}\boldsymbol{H}(\omega)\boldsymbol{U}^{-1}, & \widetilde{\boldsymbol{H}}^\sharp(\omega) &= (\boldsymbol{U}^T)^{-1}\widetilde{\boldsymbol{H}}(\omega)\boldsymbol{U}^T,\\
\boldsymbol{G}^\sharp(\omega) &= \boldsymbol{U}\boldsymbol{G}(\omega)\boldsymbol{U}^{-1}, & \widetilde{\boldsymbol{G}}^\sharp(\omega) &= (\boldsymbol{U}^T)^{-1}\widetilde{\boldsymbol{G}}(\omega)\boldsymbol{U}^T.
\end{aligned}$$

It is clear that the corresponding transformed matrix filters $\{\boldsymbol{H}_k^\sharp\}_{k\in\mathbb{Z}}$, $\{\widetilde{\boldsymbol{H}}_k^\sharp\}_{k\in\mathbb{Z}}$, $\{\boldsymbol{G}_k^\sharp\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\boldsymbol{G}}_k^\sharp\}_{k\in\mathbb{Z}}$ are given by $\boldsymbol{H}_k^\sharp = \boldsymbol{U}\boldsymbol{H}_k\boldsymbol{U}^{-1}$, $\widetilde{\boldsymbol{H}}_k^\sharp = (\boldsymbol{U}^T)^{-1}\widetilde{\boldsymbol{H}}_k\boldsymbol{U}^T$, $\boldsymbol{G}_k^\sharp = \boldsymbol{U}\boldsymbol{G}_k\boldsymbol{U}^{-1}$ and $\widetilde{\boldsymbol{G}}_k^\sharp = (\boldsymbol{U}^T)^{-1}\widetilde{\boldsymbol{G}}_k\boldsymbol{U}^T$ respectively.

Definition 1. A BMF possesses GMPs with a GMP order (d_1, d_2) if its refinement masks $\boldsymbol{H}(0)$ and $\widetilde{\boldsymbol{H}}(0)$, and its sets of equivalent scalar lowpass filters, $\mathcal{H}_\nu^\sharp(\omega)$ and $\widetilde{\mathcal{H}}_\nu^\sharp(\omega)$, $\nu = 1, 2$, associated with $\{\boldsymbol{H}_k^\sharp\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\boldsymbol{H}}_k^\sharp\}_{k\in\mathbb{Z}}$, respectively, possess the following properties:

1. Both $\boldsymbol{H}(0)$ and $\widetilde{\boldsymbol{H}}(0)$ have a *common* right 1-eigenvector,
2. $\dfrac{d^\ell}{d\omega^\ell}\mathcal{H}_\nu^\sharp(\pi) = 0, \quad \ell = 0, 1, \ldots, d_1 - 1,$

3. $\dfrac{d^\ell}{d\omega^\ell}\widetilde{\mathcal{H}}_\nu^\sharp(\pi) = 0, \quad \ell = 0, 1, \ldots, d_2 - 1,$

for all $\nu = 1, 2$, and $d_1, d_2 \geq 1$. The transition matrix $\boldsymbol{U}$ is then determined by the property that the vector $\boldsymbol{U}\widehat{\boldsymbol{\phi}}(0)$ is parallel to the vector $[1,1]^T$.

In fact, the above definition for good multiwavelet filters follows from the fact that both $\mathcal{H}_\nu^\sharp(\omega)$ and $\widetilde{\mathcal{H}}_\nu^\sharp(\omega)$ play the role of lowpass filters. Obviously, the above condition ensures that at least locally constant inputs will produce locally constant outputs. Such a property is critical for achieving good energy compaction when the multiwavelet filters are used to decompose an image ([16], [18]).

4 Construction of SABMFs

We begin with some basic properties of SABMFs. Two methods for the construction of SABMFs will then be presented. Several examples of SABMFs with different lengths are also given.

For our following discussion, we need to refer to two special matrices:

$$\boldsymbol{J} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{S} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For SABMFs, it is easy to see that $\widehat{\boldsymbol{\phi}}(0)$ is parallel to $[1,0]^T$, and thus one can use this to determine (see GMP definition) the required orthogonal matrix $\boldsymbol{U}$ for transforming $\boldsymbol{H}(\omega)$ to $\boldsymbol{H}^\sharp(\omega)$. For our purpose, we fix it as

$$\boldsymbol{U} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

4.1 Some Properties of SABMFs

We say that the finite length matrix sequences $\{\boldsymbol{H}_k\}_{k=N^\ell}^{N^u}$ and $\{\widetilde{\boldsymbol{H}}_k\}_{k=\widetilde{N}^\ell}^{\widetilde{N}^u}$ satisfy **Condition SA** if the following conditions hold:

$$\boldsymbol{H}_k = \boldsymbol{S}\boldsymbol{H}_{N^u+N^\ell-k}\boldsymbol{S}, \quad \text{for} \quad k = N^\ell, \ldots, N^u. \tag{10}$$

$$\widetilde{\boldsymbol{H}}_k = \boldsymbol{S}\widetilde{\boldsymbol{H}}_{\widetilde{N}^u+\widetilde{N}^\ell-k}\boldsymbol{S}, \quad \text{for} \quad k = \widetilde{N}^\ell, \ldots, \widetilde{N}^u. \tag{11}$$

$$\boldsymbol{H}(0) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}(0) = \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{\lambda} \end{bmatrix}, \quad \text{where} \quad |\lambda|, |\widetilde{\lambda}| < 1. \tag{12}$$

Here (10) and (11) are required for the symmetry and antisymmetry of the multiscaling functions [1] which can be expressed in the following form:

$$\phi_k(x) = (-1)^{k-1}\phi_k(N^u + N^\ell - x), \quad \widetilde{\phi}_k(x) = (-1)^{k-1}\widetilde{\phi}_k(\widetilde{N}^u + \widetilde{N}^\ell - x), \quad k = 1, 2. \tag{13}$$

Equation (12) is required for the existence of the multiscaling functions [2].

Condition SA implies that the following statements hold:

$$\boldsymbol{H}(0)[1,0]^T = [1,0]^T; \quad \widetilde{\boldsymbol{H}}(0)[1,0]^T = [1,0]^T. \tag{14}$$

$$\boldsymbol{H}_k = \boldsymbol{S}\boldsymbol{H}_{N^u+N^\ell-k}\boldsymbol{S} \Longleftrightarrow \boldsymbol{H}_k^\sharp = \boldsymbol{J}\boldsymbol{H}_{N^u+N^\ell-k}^\sharp\boldsymbol{J}. \tag{15}$$

$$\widetilde{\boldsymbol{H}}_k = \boldsymbol{S}\widetilde{\boldsymbol{H}}_{\widetilde{N}^u+\widetilde{N}^\ell-k}\boldsymbol{S} \Longleftrightarrow \widetilde{\boldsymbol{H}}_k^\sharp = \boldsymbol{J}\widetilde{\boldsymbol{H}}_{\widetilde{N}^u+\widetilde{N}^\ell-k}^\sharp\boldsymbol{J}. \tag{16}$$

Corresponding to (12), the frequency responses $\boldsymbol{H}^\sharp(\omega)$ and $\widetilde{\boldsymbol{H}}^\sharp(\omega)$ at $\omega = 0$ are given by

$$\boldsymbol{H}^\sharp(0) = \frac{1}{2}\begin{bmatrix} 1+\lambda & 1-\lambda \\ 1-\lambda & 1+\lambda \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}^\sharp(0) = \frac{1}{2}\begin{bmatrix} 1+\widetilde{\lambda} & 1-\widetilde{\lambda} \\ 1-\widetilde{\lambda} & 1+\widetilde{\lambda} \end{bmatrix}. \tag{17}$$

Equations (14) and (17) lead to the following result:

Proposition 1. *Any BMF satisfying Condition SA possesses a GMP order of at least* $(1,1)$ *if and only if both* $\boldsymbol{H}(0)$ *and* $\widetilde{\boldsymbol{H}}(0)$ *are singular, i.e.,* $\lambda = \widetilde{\lambda} = 0$.

The important question now is: how do one construct a BMF satisfying Condition SA which has a GMP order of at least $(1,1)$? Equations (15) and (16) imply that the equivalent scalar filters $\mathcal{H}_1^\sharp(\omega)$ and $\mathcal{H}_2^\sharp(\omega)$ are time reversed versions of one another; so are $\widetilde{\mathcal{H}}_1^\sharp(\omega)$ and $\widetilde{\mathcal{H}}_2^\sharp(\omega)$. The resultant matrix lowpass sequences $\{\boldsymbol{H}_k^\sharp\}_{k=N^\ell}^{N^u}$ and $\{\widetilde{\boldsymbol{H}}_k^\sharp\}_{k=\widetilde{N}^\ell}^{\widetilde{N}^u}$ have the form

$$\boldsymbol{H}_k^\sharp = \begin{bmatrix} a_{2k} & a_{2k+1} \\ a_{2(N^\ell+N^u-k)+1} & a_{2(N^\ell+N^u-k)} \end{bmatrix}, \quad k = N^\ell, \ldots, N^u, \tag{18}$$

and

$$\widetilde{\boldsymbol{H}}_k^\sharp = \begin{bmatrix} \widetilde{a}_{2k} & \widetilde{a}_{2k+1} \\ \widetilde{a}_{2(\widetilde{N}^\ell+\widetilde{N}^u-k)+1} & \widetilde{a}_{2(\widetilde{N}^\ell+\widetilde{N}^u-k)} \end{bmatrix}, \quad k = \widetilde{N}^\ell, \ldots, \widetilde{N}^u. \tag{19}$$

From the PR condition (5), the following conditions on scalar sequences $\{a_k\}_{k=2N^\ell}^{2N^u+1}$ and $\{\widetilde{a}_k\}_{k=2\widetilde{N}^\ell}^{2\widetilde{N}^u+1}$ can be derived:

$$\sum a_k\widetilde{a}_{k+4i} = 2\delta_i, \quad i \in \mathbb{Z}, \tag{20}$$

$$\sum a_k\widetilde{a}_{2\widetilde{N}^\ell+(2\widetilde{N}^u+1)-k-4i} = 0, \quad i \in \mathbb{Z}, \tag{21}$$

$$\sum a_k\widetilde{a}_{2\widetilde{N}^\ell+(2\widetilde{N}^u+1)-k+4i} = 0, \quad i \in \mathbb{Z}. \tag{22}$$

Clearly if two scalar sequences satisfy Conditions (20)-(22), then the corresponding matrix CQF can be constructed via (18) and (19).

Using (18)-(19) and Proposition 1, we see that if the scalar sequences also satisfy

$$\sum_k a_{2k} = \sum_k a_{2k+1} = 1 \text{ and } \sum_k \widetilde{a}_{2k} = \sum_k \widetilde{a}_{2k+1} = 1, \tag{23}$$

then the corresponding BMF possesses GMP order of at least (1,1).

In the following, we will study two methods for constructing SABMFs. The first method constructs SABMFs from even-length, linear-phase biorthogonal scalar wavelets while the second method constructs SABMFs from non-symmetric scalar sequences which are not necessarily related to any biorthogonal scalar wavelets. For both methods, corresponding highpass filters $\{\boldsymbol{G}_k\}_{k=M^\ell}^{M^u}$ and $\{\widetilde{\boldsymbol{G}}_k\}_{k=\widetilde{M}^\ell}^{\widetilde{M}^u}$ are constructed either directly from the lowpass filters or from solving the PR equations. Since symmetry/antisymmetry is also desired for the highpass filters, we require the highpass sequences to satisfy

$$\boldsymbol{G}_k = \boldsymbol{S}\boldsymbol{G}_{M^u+M^\ell-k}\boldsymbol{S}, \quad \text{for} \quad k = M^\ell, \ldots, M^u, \tag{24}$$

$$\widetilde{\boldsymbol{G}}_k = \boldsymbol{S}\widetilde{\boldsymbol{G}}_{\widetilde{M}^u+\widetilde{M}^\ell-k}\boldsymbol{S}, \quad \text{for} \quad k = \widetilde{M}^\ell, \ldots, \widetilde{M}^u, \tag{25}$$

which are similar to that required for the lowpass filters.

The following theorem shows that we can always associate two parameters with these highpass filters for the purpose of filter optimization.

Theorem 1. *Suppose that $\{\boldsymbol{H}_k\}$ and $\{\widetilde{\boldsymbol{H}}_k\}$ are finite length lowpass filters of a SABMF, and that the corresponding finite length highpass filters $\{\boldsymbol{G}_k\}$ and $\{\widetilde{\boldsymbol{G}}_k\}$ satisfy* (24) *and* (25). *Then the sequences $\{\boldsymbol{G}_k^{\triangleright}\}$ and $\{\widetilde{\boldsymbol{G}}_k^{\triangleright}\}$ where*

$$\boldsymbol{G}_k^{\triangleright} = \begin{bmatrix} \tau & 0 \\ 0 & \delta \end{bmatrix} \boldsymbol{G}_k, \quad \widetilde{\boldsymbol{G}}_k^{\triangleright} = \begin{bmatrix} 1/\tau & 0 \\ 0 & 1/\delta \end{bmatrix} \widetilde{\boldsymbol{G}}_k, \quad \tau, \delta \in \mathbb{R} \setminus \{0\},$$

also satisfy (24) *and* (25), *and that they too are highpass filters for the same lowpass filters $\{\boldsymbol{H}_k\}$ and $\{\widetilde{\boldsymbol{H}}_k\}$.*

The proof is straight forward and involves only verifying the PR conditions. Note that modifying the highpass sequences this way does not change its symmetric and antisymmetric properties, which means that linear phase is also unaffected.

Given that the lowpass and highpass filters of a SABMF are constructed, the two real parameters, τ and δ, provide additional flexibility for tailoring the SABMF to specific applications. For image compression, the values of these two parameters can be determined by optimizing the objective function

$$E(p,s,\alpha,\widetilde{p},\widetilde{s},\widetilde{\alpha}) = \alpha\sum_{\nu=1}^{2}\int_{0}^{s}|\mathcal{G}_\nu^\sharp(\omega)|^2 d\omega + \widetilde{\alpha}\sum_{\nu=1}^{2}\int_{0}^{\widetilde{s}}|\widetilde{\mathcal{G}}_\nu^\sharp(\omega)|^2 d\omega$$
$$+ \quad (1-\alpha)\sum_{\nu=1}^{2}\int_{p}^{\pi}((1-|\mathcal{G}_\nu^\sharp(\omega)|)^2 d\omega + (1-\widetilde{\alpha})\sum_{\nu=1}^{2}\int_{\widetilde{p}}^{\pi}((1-|\widetilde{\mathcal{G}}_\nu^\sharp(\omega)|)^2 d\omega, \quad (26)$$

where $0 \le \alpha, \widetilde{\alpha} \le 1$, $s, \widetilde{s}$ denote the stopband, and $p, \widetilde{p}$ denote the passband. The objective function corresponds to minimizing the deviation of the magnitude response from the ideal brick-wall highpass filter response. The motivation of using this objective function is the good frequency selectivity of the brick-wall filter. This filter optimization technique has shown to further improve on the image compression performance of several SABMFs, as will be shown later.

4.2 Construction of SABMFs From Linear-Phase Biorthogonal Scalar Wavelets

Proposition 2. *Any even-length, linear-phase biorthogonal scalar wavelet filter can be used to generate the lowpass and highpass sequences of a SABMF.*

Proof. Let $\{c_k\}_{k=M^\ell}^{M^u}$ and $\{\widetilde{c}_k\}_{k=\widetilde{M}^\ell}^{\widetilde{M}^u}$ be the analysis and synthesis scalar lowpass filters associated with an even-length, linear-phase biorthogonal scalar wavelet. Clearly, both $M^\ell + M^u$ and $\widetilde{M}^\ell + \widetilde{M}^u$ must be odd integers. We can assume, without loss of generality (see [3]), that $M^\ell = 2m^\ell$, $M^u = 2m^u + 1$, $\widetilde{M}^\ell = 2\widetilde{m}^\ell$ and $\widetilde{M}^u = 2\widetilde{m}^u + 1$.

To each scalar sequence, we add two zeros to the beginning and two zeros to the end. Thus we have the expanded scalar sequences: $\{c_k\}_{k=M^\ell-2}^{M^u+2}$ and $\{\widetilde{c}_k\}_{k=\widetilde{M}^\ell-2}^{\widetilde{M}^u+2}$. Using these sequences, we construct

$$\boldsymbol{H}_k^\sharp = \begin{bmatrix} c_{2k} & c_{2k+1} \\ c_{2k-2} & c_{2k-1} \end{bmatrix}, \quad k = m^\ell, \ldots, m^u + 1, \tag{27}$$

and

$$\widetilde{\boldsymbol{H}}_k^\sharp = \begin{bmatrix} \widetilde{c}_{2k} & \widetilde{c}_{2k+1} \\ \widetilde{c}_{2k-2} & \widetilde{c}_{2k-1} \end{bmatrix} \quad k = \widetilde{m}^\ell, \ldots, \widetilde{m}^u + 1. \tag{28}$$

The symmetry of the scalar sequences implies that

$$\boldsymbol{H}_k^\sharp = \boldsymbol{J}\boldsymbol{H}^\sharp_{m^\ell+m^u+1-k}\boldsymbol{J}, \quad k = m^\ell, \ldots, m^u + 1, \quad \text{and} \tag{29}$$

$$\widetilde{\boldsymbol{H}}_k^\sharp = \boldsymbol{J}\widetilde{\boldsymbol{H}}^\sharp_{\widetilde{m}^\ell+\widetilde{m}^u+1-k}\boldsymbol{J}, \quad k = \widetilde{m}^\ell, \ldots, \widetilde{m}^u + 1. \tag{30}$$

Through (15) and (16), we see that (10) and (11) are satisfied. Furthermore the fact that our scalar sequences are lowpass filters of scalar wavelets means that (23) is satisfied, and thus (12) holds for $\lambda = \tilde{\lambda} = 0$. As such, the constructed matrix CQF satisfies Condition SA.

Next, the associated highpass filters $\{\boldsymbol{G}_k\}$ and $\{\widetilde{\boldsymbol{G}}_k\}$ which satisfy (24) and (25) can be constructed via

$$\begin{aligned} \boldsymbol{G}_k &= \boldsymbol{J}\widetilde{\boldsymbol{H}}_{\widetilde{m}^\ell+\widetilde{m}^u+k}\boldsymbol{J}, \quad k = -\widetilde{m}^u, \ldots, 1-\widetilde{m}^\ell, \quad \text{and} \\ \widetilde{\boldsymbol{G}}_k &= \boldsymbol{J}\boldsymbol{H}_{m^\ell+m^u+k}\boldsymbol{J}, \quad k = -m^u, \ldots, 1-m^\ell. \end{aligned} \tag{31}$$

It is easy to verify that the constructed matrix lowpass and highpass filters satisfy the PR conditions.

□

Method 1.

Based on Proposition 2, a step-by-step procedure to construct a class of SABMFs is given as follows:

- **Step 1:** Input an even-length, linear-phase biorthogonal scalar filters $\{c_k\}_{k=M^\ell}^{M^u}$ and $\{\widetilde{c}_k\}_{k=\widetilde{M}^\ell}^{\widetilde{M}^u}$.
- **Step 2:** Construct $\{\boldsymbol{H}_k^\sharp\}_{k=m^\ell}^{m^u+1}$ and $\{\widetilde{\boldsymbol{H}}_k^\sharp\}_{k=\widetilde{m}^\ell}^{\widetilde{m}^u+1}$ using (27) and (28).
- **Step 3:** Compute the matrix lowpass filters $\{\boldsymbol{H}_k\}_{k=m^\ell}^{m^u+1}$ and $\{\widetilde{\boldsymbol{H}}_k\}_{k=\widetilde{m}^\ell}^{\widetilde{m}^u+1}$ using $\boldsymbol{H}_k = \boldsymbol{U}^{-1}\boldsymbol{H}_k^\sharp\boldsymbol{U}$ and $\widetilde{\boldsymbol{H}}_k = \boldsymbol{U}^T\widetilde{\boldsymbol{H}}_k^\sharp\boldsymbol{U}^{T^{-1}}$.
- **Step 4:** Compute the matrix highpass filters $\{\boldsymbol{G}_k\}_{k=-m^u}^{1-m^\ell}$ and $\{\widetilde{\boldsymbol{G}}_k\}_{k=-\widetilde{m}^u}^{1-\widetilde{m}^\ell}$ using (31).

An example illustrating this procedure can be found in Appendix 1.

It is important to point out that linear-phase, **odd-length** biorthogonal scalar wavelets such as the popular Daubechies' D(9/7) [3] filter **cannot** be used in a similar manner to construct SABMFs. Nevertheless, we can construct a BMF which does not satisfy Condition SA from any odd-length scalar wavelet. The construction of such BMFs are however beyond the scope of this paper.

4.3 Construction of Other SABMFs Possessing GMPs

More general SABMFs which are not linked to linear phase biorthogonal scalar wavelets can be constructed by solving for *non-symmetric* scalar sequences $\{a_k\}$ and $\{\widetilde{a}_k\}$ which satisfy (20)-(22) and which corresponding matrix lowpass filters $\{\boldsymbol{H}_k\}$ and $\{\widetilde{\boldsymbol{H}}_k\}$ have their refinement masks in the form (17) with $\lambda = \widetilde{\lambda} = 0$. Note that these scalar sequences need not generate a scalar wavelet. The resultant multiwavelet filter has GMP order at least (1,1) and approximation order at least (1,1), and invariably has a number of free parameters, more for filters of longer lengths. Such free parameters are useful for filter design/optimization purposes. We will focus on the use of these parameters for improving image compression.

Method 2.

A step-by-step procedure to construct a SABMF with GMP order (d_1, d_2) and approximation order (p_1, p_2) is given in the following.

- **Step 1:** Let scalar sequences $\{a_k\}_{k=2N^\ell}^{2N^u+1}$ and $\{\widetilde{a}_k\}_{k=2\widetilde{N}^\ell}^{2\widetilde{N}^u+1}$ be related to the matrix lowpass sequences via (18) and (19). Construct the system of nonlinear equations from (20)-(22) and (17) with $\lambda = \widetilde{\lambda} = 0$.
- **Step 2:** Augment the system with equations for higher GMP order (d_1, d_2) using Definition 1.
- **Step 3:** Augment the system with equations for higher approximation order (p_1, p_2) using (8).
- **Step 4:** Seek a solution or solutions of the nonlinear equations using symbolic packages such as Mathematica [21] or Maple V [6].
- **Step 5:** Solve the PR equations (5)-(7) for the corresponding matrix highpass sequences. Apply Theorem 1 to introduce the two highpass parameters.

A good strategy for choosing d_1, d_2, p_1, p_2 for a BMF of specific lengths is to pick their values such that the resultant matrix lowpass sequences still has one free parameter remain for further optimization with regards to having the lowpass magnitude response as close as possible to the ideal brick-wall filter.

Following the above procedure, a large number of SABMFs of varying lengths have been constructed. Of these, the relatively short length 4/4 and 5/5 filter families are presented in details in Appendices 2 and 3. They are chosen not just for ease of exposition but also for their better compression performance with relatively lower cost of computation when compared to the popular D(9/7) and V(10/18) scalar filters.

SABMFs of longer lengths have also been constructed. After applying the filter optimization process for good image compression, we have obtained a number of these with better compression performance than both D(9/7) and V(10/18) filters. For convenience of exposition, we will present only three filters, BSA(6/6)*, BSA(8/6)* and BSA(7/9)* with their filter coefficients given, respectively, in Tables 1-3.

We have numerically verified that the transition operators of all the above-mentioned SABMFs satisfy Condition E, and as such each of them generates a BMWS.

Method 2 involves solving a system of nonlinear equations for the matrix lowpass and highpass sequences of a SABMF. The difficulty level of such a task increases with the lengths of the filter sought, more so if the solution desired is to be in symbolic form (in contrast to being numeric) such as the above-mentioned length 4/4 and 5/5 filters. Symbolic algebra packages, such as Maple V, are able to provide symbolic solutions for up to length 6/6 with ease and time efficiency. But beyond this, it is more effective to adopt an alternative means of obtaining parameterized filter solutions, namely, the lifting scheme proposed by Goh *et al* [5]. BSA(8/6)* and BSA(7/9)* were obtained from parameterized solutions generated using the lifting scheme and then with the parameters utilized for achieving GMPs, and closest proximity to the brick-wall filters.

5 Application of Multiwavelets to Image Compression

This section briefly explains how the designed SABMFs can be applied for multiscale signal decomposition and reconstruction. It should be noted that Mallat's multiresolution algorithm [10] for scalar wavelets cannot be used directly for multiwavelet filters, each of which requires a vectorized input signal. The problem of obtaining the vector input streams from a given signal is known as *multiwavelet initialization* or *pre-filtering.* In [18], we proposed a framework for multiwavelet initialization which works for any multiwavelet filter and gives a compact representation of the input signal. Readers who are interested in a more detailed explanation of the proposed pre-filtering may refer to [18].

5.1 Multiwavelet Decomposition and Reconstruction Algorithms

Denote $\boldsymbol{\phi}_{\ell,k} = [\phi_{1,\ell,k}, \phi_{2,\ell,k}]^T$ and $\widetilde{\boldsymbol{\phi}}_{\ell,k} = [\widetilde{\phi}_{1,\ell,k}, \widetilde{\phi}_{2,\ell,k}]^T$ where $\phi_{\nu,\ell,k} = \sqrt{2}\phi_\nu(2^\ell x - k)$ and $\widetilde{\phi}_{\nu,\ell,k} = \sqrt{2}\phi_\nu(2^\ell x - k)$, $\nu = 1, 2$. Likewise, we define $\boldsymbol{\psi}_{\ell,k}$, $\widetilde{\boldsymbol{\psi}}_{\ell,k}$ and their component functions $\psi_{\nu,\ell,k}$ and $\widetilde{\psi}_{\nu,\ell,k}$, $\nu = 1, 2$. Since $\widetilde{V}_\ell \subset \widetilde{V}_{\ell+1}$ for $\ell \in \mathbb{Z}$ and $\overline{\bigcup_{\ell\in\mathbb{Z}} \widetilde{V}_\ell} = L^2(\mathbb{R})$, for sufficiently large ℓ, say L, we can assume that a given signal $f \in \widetilde{V}_L$. Thus we have

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} \boldsymbol{p}_{L,k}^T \widetilde{\boldsymbol{\phi}}_{L,k}(x) \\ &= \sum_{k\in\mathbb{Z}} \boldsymbol{p}_{L_0,k}^T \widetilde{\boldsymbol{\phi}}_{L_0,k}(x) + \sum_{L_0\le\ell<L} \sum_{k\in\mathbb{Z}} \boldsymbol{q}_{\ell,k}^T \widetilde{\boldsymbol{\psi}}_{\ell,k}(x), \end{aligned} \tag{32}$$

where $L_0 < L$. Let

$$p_{\nu,\ell,k} = \int f(x)\phi_{\nu,\ell,k}(x)dx, \quad q_{\nu,\ell,k} = \int f(x)\psi_{\nu,\ell,k}(x)dx, \quad \nu = 1, 2,$$

and

$$\boldsymbol{p}_{\ell,k} = [p_{1,\ell,k}, \quad p_{2,\ell,k}]^T, \quad \boldsymbol{q}_{\ell,k} = [q_{1,\ell,k}, \quad q_{2,\ell,k}]^T.$$

By equations (1)–(4), one can derive the decomposition algorithm

$$\boldsymbol{p}_{\ell-1,k} = \sum_{m\in\mathbb{Z}} \boldsymbol{H}_{m-2k}\boldsymbol{p}_{\ell,m}, \tag{33}$$

$$\boldsymbol{q}_{\ell-1,k} = \sum_{m\in\mathbb{Z}} \boldsymbol{G}_{m-2k}\boldsymbol{p}_{\ell,m}, \quad \ell = L, L-1, \ldots, L_0+1, \tag{34}$$

as well as the reconstruction algorithm

$$\boldsymbol{p}_{\ell,k} = \sum_{m\in\mathbb{Z}} \widetilde{\boldsymbol{H}}^T_{k-2m}\boldsymbol{p}_{\ell-1,m} + \sum_{m\in\mathbb{Z}} \widetilde{\boldsymbol{G}}^T_{k-2m}\boldsymbol{q}_{\ell-1,m}, \quad \ell = L_0+1, \ldots, L. \tag{35}$$

The above decomposition and reconstruction algorithms can be realized as a multiwavelet filter bank. Figure 1 depicts a 1-level subband decomposition and reconstruction framework for discrete multiwavelet transform. The left half of the figure represents a 1-level multiwavelet decomposition, where a vector input stream is decomposed by a matrix lowpass filter, $\boldsymbol{H}$, and a matrix highpass filter, $\boldsymbol{G}$, to generate the next lower resolution. This is followed by subsampling by a factor of 2 to preserve compact representation of the input signal in a 2-band filtering process. For octave-bandwidth decomposition, only the lowpass subbands will be decomposed iteratively to produce subsequent lower resolutions. Graphically, a N-level decomposition will consist of a cascade of N such 1-level decompositions, each operating on the lowpass subbands of the previous resolution. The right half of Figure 1 represents the corresponding 1-level multiwavelet reconstruction. The subbands are first upsampled by a factor of 2 before they are filtered by the synthesis matrix filters to recover the original vector stream. An obvious question now is how to obtain the vectorized input stream from a given one-dimensional (scalar) input signal (e.g. a row or column of an image). This problem of multiwavelet initialization will be dealt with in the next subsection.

5.2 Multiwavelet Pre-filter and Post-filter

Given a signal $f(x) \in L^2(\mathbb{R})$, we can expand $f(x)$ by means of (32) with vector-valued coefficients $\boldsymbol{p}_{L,k}$ which can then be input to the first level decomposition part of the multiwavelet filter bank described in the previous section. A good criterion to construct an efficient pre-filter, for evaluation of the initial coefficients, is to take into account the nature of the components of multiscaling functions we are dealing with. More accurately, the pre-filter is determined by the 1-eigenvector of the refinement mask $\boldsymbol{H}(0)$. Based on the analysis in Section 3, a simple, orthogonal and non-redundant pre-filter can be formulated [18]. This pre-filter is represented by an orthogonal matrix

$$\boldsymbol{M} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \text{or} \quad \boldsymbol{M} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}, \tag{36}$$

where $\theta = -\pi/4$ when $\widehat{\phi}_1(0) = 0$, otherwise $\theta = \pi/4 - \tan^{-1}(\widehat{\phi}_2(0)/\widehat{\phi}_1(0))$ when $\theta = (-\pi/4, \pi/4]$. The matrix $\boldsymbol{M}$ has the property that vector $\boldsymbol{M}[1,1]^T$ is a constant multiple of the 1-eigenvector of $\boldsymbol{H}(0)$. Suppose that the input signal $f(x)$ is sampled to give an even-length sequence, $y_0, \ldots, y_{n-1}$, then the initial data stream $\boldsymbol{p}_{L,k}$ is obtained by

$$\boldsymbol{p}_{L,k} = \boldsymbol{M} \begin{bmatrix} y_{2k} \\ y_{2k+1} \end{bmatrix}. \tag{37}$$

A natural question now arises as to which of the two forms for $\boldsymbol{M}$ in (36) should be used for a given multiwavelet filter. Note that since $\boldsymbol{M}^{-1}[\widehat{\phi}_1(0), \widehat{\phi}_2(0)]^T$ is parallel to vector $[1,1]^T$, and $\mathcal{H}^\sharp_\nu(\omega)$, $\nu = 1, 2$, are the equivalent scalar filters of $\boldsymbol{M}^{-1}\boldsymbol{H}_k\boldsymbol{M}$, we can evaluate the measure of deviation from the ideal "brick-wall" lowpass filter

$$E = \sum_{\nu=1}^{2} \int_0^{\pi/2} (1 - |\mathcal{H}^\sharp_\nu(\omega)|)^2 d\omega + \int_{\pi/2}^{\pi} |\mathcal{H}^\sharp_\nu(\omega)|^2 d\omega,$$

and then select the matrix $\boldsymbol{M}$ which gives the smaller value of E. This criterion of selecting the pre-filter matrix $\boldsymbol{M}$ is very dependent on the given multiwavelet filter. All SABMFs have $[\widehat{\phi}_1(0), \widehat{\phi}_2(0)]$ parallel to $[1,0]$, and this implies that $\theta = \pi/4$ in (36). For all SABMFs presented in Section 4, the first form in (36) is found to produce smaller E. Thus for these SABMFs, one should use

$$\boldsymbol{M} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{38}$$

It should be cautioned that there are other SABMFs which require the use of the second form of $\boldsymbol{M}$.

In order to recover the original signal (without any quantization), post-filtering is needed after applying the multiwavelet reconstruction algorithm. Let $\mathcal{Q}$ denote the post-filtering multirate matrix filter which is associated with the pre-filter $\boldsymbol{M}$ such that $\mathcal{Q}\boldsymbol{M} = \boldsymbol{I}$. Thus $\mathcal{Q} = \boldsymbol{M}^{-1}$.

Figure 2 illustrates both the pre-filter and post-filter that are integrated into the multiwavelet decomposition and reconstruction algorithms (shown in Figure 1) to produce the proposed generalized framework for discrete multiwavelet transform. From the block diagram of the pre-filter, it is illustrative how the scalar input stream, $\boldsymbol{y}_k$, is appropriately downsampled and delayed to generate a vector stream, consisting of the even- and odd-sampled sequences, $\boldsymbol{y}_{2k}$ and $\boldsymbol{y}_{2k+1}$. This vectorized stream is then pre-filtered with the matrix filter $\boldsymbol{M}$ to produce the desired vector input stream representative of the input signal, as expressed in (37). It is also noted that the downsampling is necessary for preserving a compact (non-redundant) representation of the original signal during pre-filtering. The vector input stream can now be fed to the inputs of Figure 1 for multiresolution decomposition. In a similar manner, the output streams of Figure 1 after reconstruction can be post-filtered to recover the original scalar stream. The block diagram of the post-filter in Figure 2 describes how this is carried out by appropriately upsampling, delaying, and combining the components of the vector stream into a scalar stream.

5.3 Application to 2D Image

Using the same idea of separable decomposition along each dimension of a 2-D image, the above multirate pre-filter and post-filter can now be integrated with Mallat's pyramid algorithms [10], where tensor products of the 1-D filter banks are used to process 2-D images. Figure 3 shows the multiwavelet framework for image decomposition. The pre-filter is first applied to all the rows of the image, before the first level decomposition is applied to each of the resultant rows. This process is then repeated to the columns. At the end of this first level of 2-D multiwavelet decomposition, we have a 16-subband intermediate image. All subsequent levels of 2-D multiwavelet decomposition are applied only to the four subbands in the upper left corner of the current intermediate image. No pre-filtering is performed for these later 2-D decompositions where the 1-D multiwavelet decompositions are applied first to the rows followed by those to the columns. In this fashion, a L-level decomposition of a 2-D image will produce $4(3L+1)$ subbands. The 2-D reconstruction of a 2-D image is obtained by simply performing all the steps described above for decomposition in the reverse order.

6 Performance Analysis of Wavelet Filters

In this section, we investigate the performance of various scalar and multiwavelet filters for image compression. For the purpose of fair and consistent comparisons, we have chosen one of the best wavelet-based image codecs called "Set Partitioning in Hierarchical Trees" (SPIHT), as proposed by Said and Pearlman [12]. We found that other codecs such as [17] have also demonstrated comparable relative performance improvements. Since the proposed SABMFs are symmetric or antisymmetric, it was shown in [23] that better results can be achieved by employing symmetric extension at the boundaries of the image, as compared to using periodic extension. The simulations were carried out using a large number of images and at various compression ratios. Due to the limitation of space, we have selected two standard grayscale images, namely, Lena and Barbara, as representative test images. Lena is chosen because of its predominantly "smooth" background which is typical of most natural images. On the other hand, Barbara is selected for its high-frequency or texture regions at the table cloth and trousers.

In order to provide an insightful analysis, both the objective measure of peak signal-to-noise (PSNR) ratio and subjective visual quality of reconstructed images are presented. In addition, it is also observed that a longer wavelet filter that has more vanishing moments will generally perform better than a shorter filter, but this usually comes with a higher computational cost. Therefore, we also take into account the computational complexity of each wavelet filter when performing the comparisons.

6.1 Computational Cost of Wavelet Filters

One important consideration in application is the computational complexity of applying a given filter, which directly relates to its implementational efficiency. For most practical applications, computational complexity is governed by the number of multiplications and additions involved, with the former being the dominating factor. Here we will use these operation counts as a measure of the computational cost of using a wavelet filter. In the context of wavelet decomposition (or reconstruction), these counts are generally directly proportional to the sum of the lengths of lowpass and highpass filters. However, it is possible to further reduce the multiplication count if we can exploit any symmetry in the filters.

6.1.1 Scalar Wavelet Filter System

Consider the computational cost of a L-level decomposition (or reconstruction) of an $M \times N$ image using an octave bandwidth structure. Since each level is subsampled by a factor of two along the rows and columns, the number of multiplications and additions using a scalar filterbank system with lowpass and highpass filter lengths[2] of F_ℓ and F_h, respectively, is given by the tuple

$$(\#mult, \#add) = \big((F_\ell + F_h)\mathcal{L}, (F_\ell + F_h - 2)\mathcal{L}\big) \tag{39}$$

where $\mathcal{L} = MN\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{L-1}}\right)$.

For a linear-phase biorthogonal scalar wavelet, however, we can reduce the multiplication cost by half as the lowpass and highpass filters are symmetrical. In general, the computational cost of any symmetric/antisymmetric scalar filter can be expressed as

$$(\#mult, \#add) = \left(\left(\left\lfloor \frac{F_\ell + 1}{2} \right\rfloor + \left\lfloor \frac{F_h + 1}{2} \right\rfloor\right)\mathcal{L}, (F_\ell + F_h - 2)\mathcal{L}\right), \tag{40}$$

where $\lfloor x \rfloor$ denotes the largest integer than is smaller than x. The cost can be further lowered if there are embedded zeros, one's, or entries with identical magnitude in each filter sequence.

6.1.2 Multiwavelet Filter System

Similarly let F_ℓ and F_h denote the number of non-zero filter coefficients of the matrix lowpass and highpass filters, respectively. Also denote F_p as the number of non-zero matrix coefficients of the pre-filter (or post-filter). In general, the number of multiplications and additions required for a L-level octave-bandwidth multiresolution decomposition (or reconstruction) of an $M \times N$ image, including pre-filtering (or post-filtering), is given in (#mult,#add) form by

$$\begin{aligned} & \text{Cost of pre-filtering} + \text{Cost of } L\text{-level image decomposition} \\ = \quad & \big(F_p MN, (F_p - 2)MN\big) + \left(\frac{F_\ell + F_h}{2}\mathcal{L}, \left(\frac{F_\ell + F_h}{2} - 2\right)\mathcal{L}\right). \end{aligned} \tag{41}$$

Note that the factor of 1/2 in the second term is introduced by the non-redundant pre-filtering framework which allows a subsampling by 2, and halving of the input signal length [18]. In a similar manner, the cost component due to multiplications can also be reduced if the BMFs are symmetric/antisymmetric. We further show below that the cost of pre-filtering (or post-filtering) can in fact become negligible for the proposed class of SABMFs.

6.1.3 Comparison between scalar and multiwavelet systems

At first glance, the requirement for pre-filtering in employing a multiwavelet filter may introduce additional computations. However, it can be shown that the cost of pre- and post-filtering for all

[2] From a computational viewpoint, the *length* represents the number of non-zero filter coefficients. It is also worth noting that filter coefficients that are integer powers of two will require no multiplications but mere shift operations by the processor.

SABMFs will involve no multiplications, but only addition/subtraction operations. This is because for SABMFs, $\theta = \pi/4$ in (36) and thus it is clear that we will always have, up to a normalization constant, coefficients of the pre- and post-filters which are ± 1. Since the normalization constant (in this case, $\frac{1}{\sqrt{2}}$) can be absorbed into the *first* level of decomposition (or the *last* level of reconstruction), the computational cost of pre- and post-filtering can be ignored. By exploiting symmetry, the number of multiplications and additions required by using each wavelet filter shown in Tables 4 and 5 is given in Table 6.

From Table 6, we see that the application of BSA(4/4)* in the mulitwavelet framework described in Section 5 will demand a computational cost involving a multiplication count 2/3 of that for D(9/7), albeit the addition counts are the same. Both BSA(5/5)* and BSA(6/6)* have lower addition and multiplication counts when compared against V(10/18). More specifically, their multiplication counts are 4/7 and 11/14, respectively, of that for V(10/18). The possible speedup obtained using these SABMFs will become more critical for real-time implementation in video compression. The application of V(10/18) is also more costly than the BSA(8/6)* filter which has better compression performances.

6.2 Performance Comparisons for Image Compression

Table 4 compares the PSNR performances of the proposed Method 1 for constructing SABMFs directly from even-length linear-phase biorthogonal scalar wavelets. The SABMFs M(2/4), M(4/6) and M(6/10), were generated from the biorthogonal scalar filters, Daubechies' D(2/6), D(6/10) and Villasenor's V(10/18), respectively. These three scalar filters were among the 'best' biorthogonal scalar filter banks for image compression, as found by Villasenor *et al* [20]. It can be observed that, on the average, the derived SABMFs can perform better than their corresponding parent scalar filters for both images at all five compression ratios. More specifically, PSNR improvements of up to 0.45 dB, 0.43 dB and 0.40 dB, respectively, could be achieved for the Barbara image, while only marginal improvements of up to 0.12 dB for the Lena image. By varying the highpass optimization parameters τ and δ, we can obtain a better M(4/6)* filter with $\tau = 0.945$ and $\delta = 1.03$, and obtain M(6/10)* with $\tau = 0.975$ and $\delta = 1.146$. However, varying the highpass parameters has little impact on the compression performance for the shorter M(2/4) filter, which could mainly be due to the default values of $\tau = \delta = 1$ being near-optimal. From Table 6, we can further conclude that the SABMFs constructed using Method 1 can give improved compression performance with comparable computational costs.

The compression results of SABMFs constructed using Method 2 are portrayed in Table 5. In particular, we have organized the filters into two groups according to their relative computational costs. The first group consists of BSA(4/4)*, BSA(5/5)* and BSA(6/6)* against the popular Daubechies' D(9/7) scalar filter, and the second group comprises BSA(4/4)*, BSA(5/5)*, BSA(6/6)*, BSA(8/6)* and BSA(7/9)* against the longer Villasenor's V(10/18) scalar filter.

In the first group, all three multiwavelet filters outperform D(9/7) across the board, with greater improvements for longer multiwavelet filters, and more significant improvements for the Barbara image than for the Lena image. Specifically, although the performance gain of BSA(4/4)* over D(9/7) is only marginal for the smooth Lena image, considerable PSNR improvement of up to 0.43 dB could be obtained for the Barbara image. More importantly, such improvements in PSNR values do correspond well with better preservation of edges and textures, as depicted in Figure 4. Recall that these improvements also come with a lower computational cost. The longer BSA(5/5)* and BSA(6/6)* filters can perform better than D(9/7) by up to 0.63 dB with comparable computational complexity.

In the second group, with the exception of BSA(4/4)* and BSA(5/5)*, all the multiwavelet filters also generally outperform V(10/18) for both images at all compression ratios. Interestingly, the BSA(5/5)*, BSA(6/6)* and BSA(8/6)* filters can yield performance improvements of up to 0.71 dB with lower computational costs. The BSA(7/9)* filter also performs much better than V(10/18) but with a slightly higher cost. On the other hand, we note that BSA(4/4)* performs relatively well with only a fraction of the computational cost incurred by V(10/18).

To further illustrate the savings in computational times, we pick from the first group, BSA(4/4)* versus D(9/7) and from the second, BSA(6/6)* versus V(10/18). The typical (average) execution times for the D(9/7), V(10/18), BSA(4/4)* and BSA(6/6)* filters are 0.11s, 0.19s, 0.08s and 0.15s respectively for a six-level decomposition of a 512×512 image using the respective multiresolution algorithms on an Intel Pentium-II 400Mhz PC running Win NT. The comparison ratios of 0.727 for BSA(4/4)* over D(9/7), and 0.790 for BSA(6/6)* over V(10/18) agree well with the theoretical ratios of 0.666 and 0.786 when only the multiplication components are taken into account. The small differences can be attributed to the different addition counts for the filters, and other aspects of software implementation such as array indexing, loops and function calls.

The reasons for the better performance of the given multiwavelet filters over the two selected scalar filters include 'better' frequency responses for better energy compaction, greater regularity and approximation order of the corresponding wavelet/scaling functions and the proposed multiwavelet framework for implementating the filters. For the comparison pairs of BSA(4/4)* versus D(9/7) and BSA(6/6)* versus V(10/18), we could account for the differences in performance by examining the frequency magnitude response curves of the analysis filters[3], as shown in Figure 5. BSA(4/4)* has a response closer to that of the ideal brickwall filter in the low frequency region than for D(9/7), although the latter has a slightly faster transition to zero in high frequency region. The response curve for BSA(6/6)* is clearly more desirable than that for V(10/18).

7 Conclusions and Future Research Directions

We have successfully extended our earlier work on the construction of symmetric-antisymmetric orthonormal multiwavelet filters that possess good multifilter properties (GMPs) to the biorthogonal case. Two methods for the design of new families of symmetric-antisymmetric biorthogonal multiwavelet filters (SABMFs) were proposed. The first method allows direct construction from any even-length, linear-phase biorthogonal scalar wavelets. The second method gives a step-by-step procedure to construct more general SABMFs possessing GMPs. Extensive image compression experiments confirmed that the proposed class of SABMFs could give performance improvements over some popular biorthogonal scalar wavelets such as Daubechies' D(9/7) and Villasenor's V(10/18). In addition, such improvements were achieved with comparable or lower computational requirements.

Motivated by the observed performance improvements, the following two research areas have been identified to be promising. First, we could extend the present multiwavelet framework to a multiwavelet-packet structure. Such an extension is likely to be most suitable for compressing images with more high-frequency contents. Second, we could design a new coding scheme which better exploits the new inter-subband relationships in the present multiwavelet framework. This will definitely improve the compression further by having a more compact representation of the wavelet coefficients which are to be coded. Preprints of our work in the reference section, and more details about our past and on-going work can be obtained on the World Wide Web via "http://wavelets.math.nus.edu.sg/thamjy/".

[3]For the multiwavelet filters, the frequency magnitude response curves are obtained from their equivalent scalar filters.

References

[1] C. K. Chui, and J. Lian, "A Study on Orthonormal Multiwavelets", Appl. Numer. Math., Vol. 20, 1996, pp. 273-298.

[2] W. Dahmen and C. A. Micchelli, "Biorthogonal Wavelet Expansions", Constr. Approx., Vol. 13, No. 3, 1997, pp. 293-328.

[3] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[4] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, "Fractal Functions and Wavelet Expansions Based on Several Scaling Functions", J. Approx. Theory, Vol. 78, 1994, pp. 373-401.

[5] S. S. Goh, Q. Jiang, and T. Xia, "Construction of Biorthogonal Multiwavelets Using the Lifting Scheme", preprint, Wavelets Strategic Research Programme, National University of Singapore, 1998.

[6] K. M. Heal, M. Hansen, and K. Richard, Maple V Learning Guide for Release 5, Springer Verlag, 1997.

[7] C. Hel, G. Strang and V. Strela, "Approximation by Translates of Refinable Functions", Numer. Math., Vol. 73, 1996, pp. 75-94.

[8] T. N. T. Goodman, and S. L. Lee, "Wavelets of Multiplicity r", Trans. Amer. Math. Soc., Vol. 342, 1994, pp. 307-324.

[9] J. Lebrun, and M. Vetterli, "Balanced Multiwavelets: Theory and Design", IEEE Trans. Signal Process., Special issue on wavelets and filter banks, Vol. 46, No. 4, 1998, pp. 1119-1125.

[10] S. Mallat, "Multifrequency Channel Decompositions of Images and Wavelet Models", IEEE Trans. Acoust. Speech Signal Process., Vol. 37, 1989, pp. 2091-2110.

[11] Z. Shen, "Refinable Function Vectors", SIAM J. Math. Anal., Vol. 29, 1998, pp. 234-249.

[12] A. Said and A. Pearlman, "A New, Fast and Efficient Image Codec Based on Set Partitioning in Hierarchical Trees", IEEE Trans. on Image Process., Vol. 6, No. 3, Feb. 1997, pp. 240-250.

[13] L. X. Shen, H. H. Tan, and J. Y. Tham, "Symmetric-Antisymmetric Orthonormal Multiwavelets and Related Scalar Wavelets", preprint, Wavelets Strategic Research Programme, National University of Singapore, 1997.

[14] V. Strela, "A Note on Construction of Biorthogonal Multi-scaling Functions", Contemporary Mathematics, A. Aldroubi and E. B. Lin (eds.), AMS, 1998, pp. 149-157.

[15] G. Strang and T. Q. Nguyen, Wavelets and Filter Banks, Wellesley-Cambridge Press, 1996.

[16] V. Strela, P. N. Heller, G. Strang, P. Topiwala, and C. Heil, "The Application of Multiwavelet Filter Banks to Image Processing", IEEE Trans. on Image Process., Vol. 8, No. 4, 1999, pp.548-563.

[17] J. Y. Tham, S. Ranganath, A. K. Kassim, "Highly Scalable Wavelet-Based Video Codec for Very Low Bit Rate Environment", IEEE J. on Selected Areas in Comms., Special Issue on Very Low Bit-rate Video Coding, Vol. 16, No. 1, Jan. 1998, pp. 12-27.

[18] J. Y. Tham, L. X. Shen, S. L. Lee, and H. H. Tan, "A General Approach for Analysis and Application of Discrete Multiwavelet Transforms", to appear in IEEE Trans. on Signal Process..

[19] J. Y. Tham, L. X. Shen, S. L. Lee, and H. H. Tan, "Good Multifilter Properties: A New Tool for Understanding Multiwavelets", Proc. Conference for International Conference on Imaging, Science, Systems and Technology (CISST98), Las Vegas, U.S.A, 1998, pp.52-59.

[20] J. Villasenor, B. Belzer, and J. Liao, "Wavelet Filter Evaluation for Image Compression", IEEE Trans. on Image Process., Vol. 2, Aug. 1995, pp. 1053-1060.

[21] S. Wolfram, The Mathematica Book, 3rd Edition, Cambridge Univ Press, 1996.

[22] X. G. Xia, J. S. Geronimo, D. P. Hardin, and B. W. Suter, "Design of Prefilters for Discrete Multiwavelet Transform", IEEE Trans. Signal Process., Vol. 44, 1996, pp. 25-35.

[23] T. Xia and Q. Jiang, "Optimal Multifilter Banks: Design, Related Symmetric-Extension and Application to Image Compression", to appear in IEEE Trans. on Signal Process..

Appendix 1

Consider the biorthogonal D(2/6) scalar wavelet filter [3] with the following lowpass sequences:

$$\{\widetilde{c}_k\}_{k=-2}^{3} = \left\{-\frac{1}{8}, \frac{1}{8}, 1, 1, \frac{1}{8}, -\frac{1}{8}\right\}, \quad \{c_k\}_{k=0}^{1} = \{1, 1\}.$$

Following **Step 1-3** of Method 1, we can easily construct a length 2/4 SABMF, M(2/4), with the following transformed matrix lowpass sequences

$$\widetilde{\boldsymbol{H}}^{\sharp}_{-1} = \begin{bmatrix} -\frac{1}{8} & \frac{1}{8} \\ 0 & 0 \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}^{\sharp}_{0} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{8} & \frac{1}{8} \end{bmatrix}, \quad \boldsymbol{H}^{\sharp}_{0} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\widetilde{\boldsymbol{H}}^{\sharp}_{1} = \boldsymbol{J}\widetilde{\boldsymbol{H}}^{\sharp}_{0}\boldsymbol{J}, \quad \widetilde{\boldsymbol{H}}^{\sharp}_{2} = \boldsymbol{J}\widetilde{\boldsymbol{H}}^{\sharp}_{-1}\boldsymbol{J}, \quad \boldsymbol{H}^{\sharp}_{1} = \boldsymbol{J}\boldsymbol{H}^{\sharp}_{0}\boldsymbol{J}.$$

or, equivalently, the symmetric-antisymmetric matrix lowpass sequences

$$\widetilde{\boldsymbol{H}}_{-1} = \begin{bmatrix} 0 & \frac{1}{8} \\ 0 & -\frac{1}{8} \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}_{0} = \begin{bmatrix} 1 & \frac{1}{8} \\ -1 & \frac{1}{8} \end{bmatrix}, \quad \boldsymbol{H}_{0} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},$$

$$\widetilde{\boldsymbol{H}}_{1} = \boldsymbol{S}\widetilde{\boldsymbol{H}}_{0}\boldsymbol{S}, \quad \widetilde{\boldsymbol{H}}_{2} = \boldsymbol{S}\widetilde{\boldsymbol{H}}_{-1}\boldsymbol{S}, \quad \boldsymbol{H}_{1} = \boldsymbol{S}\boldsymbol{H}_{0}\boldsymbol{S}.$$

The corresponding matrix highpass filters are obtained from equation (31).

Appendix 2

A length 4/4 SABMF family

Using Method 2, a family of length 4/4 SABMFs with GMP order at least (3,1) is constructed with the matrix lowpass sequences given by

$$\boldsymbol{H}_{-1} = \begin{bmatrix} \frac{2\gamma}{8\gamma-3} & \frac{1}{8} \\ 2\alpha\gamma & \frac{\alpha(8\gamma-1)}{8} \end{bmatrix}, \quad \boldsymbol{H}_{0} = \begin{bmatrix} \frac{3(2\gamma-1)}{8\gamma-3} & \frac{1}{8} \\ \alpha(2\gamma-1) & -\frac{\alpha(8\gamma-1)}{8} \end{bmatrix},$$

$\boldsymbol{H}_1 = \boldsymbol{S}\boldsymbol{H}_0\boldsymbol{S}$, $\quad \boldsymbol{H}_2 = \boldsymbol{S}\boldsymbol{H}_{-1}\boldsymbol{S}$, and

$$\widetilde{\boldsymbol{H}}_{-1} = \begin{bmatrix} 2\gamma & \frac{16\gamma(2\gamma-1)}{8\gamma-3} \\ \frac{2}{3}\alpha\gamma(8\gamma-1) & \frac{16}{3}\alpha\gamma(2\gamma-1) \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}_{0} = \begin{bmatrix} 1-2\gamma & \frac{16\gamma(2\gamma-1)}{8\gamma-3} \\ -\alpha(8\gamma-1)(2\gamma-1) & -\frac{16}{3}\alpha\gamma(2\gamma-1) \end{bmatrix},$$

$\widetilde{\boldsymbol{H}}_1 = \boldsymbol{S}\widetilde{\boldsymbol{H}}_0\boldsymbol{S}$, $\quad \widetilde{\boldsymbol{H}}_2 = \boldsymbol{S}\widetilde{\boldsymbol{H}}_{-1}\boldsymbol{S}$, with γ being the parameter, and $\alpha = \sqrt{3/(8\gamma-1)/(8\gamma-3)}$. Note that when $\gamma = (17-\sqrt{241})/64$, this SABMF has a GMP order $(3, 2)$.

The corresponding matrix highpass sequences are obtained by solving the PR conditions, and they are given by

$$\boldsymbol{G}_{-1} = \begin{bmatrix} \dfrac{16\rho\gamma(2\gamma-1)}{8\gamma-3} & 2\rho\gamma \\ 2 & \dfrac{(8\gamma-5)}{4(2\gamma-1)} \end{bmatrix}, \quad \boldsymbol{G}_0 = \begin{bmatrix} \dfrac{-16\rho\gamma(2\gamma-1)}{8\gamma-3} & -\rho(6\gamma-1) \\ 2 & \dfrac{-(16\gamma^2-18\gamma+3)}{8\gamma(2\gamma-1)} \end{bmatrix},$$

$\boldsymbol{G}_1 = \boldsymbol{S}\boldsymbol{G}_0\boldsymbol{S}, \quad \boldsymbol{G}_2 = \boldsymbol{S}\boldsymbol{G}_{-1}\boldsymbol{S}$, and

$$\widetilde{\boldsymbol{G}}_{-1} = \begin{bmatrix} \dfrac{1}{8\rho} & 2 \\ \dfrac{\gamma(2\gamma-1)}{8\gamma-3} & \dfrac{16\rho\gamma(2\gamma-1)}{8\gamma-5} \end{bmatrix}, \quad \widetilde{\boldsymbol{G}}_0 = \begin{bmatrix} \dfrac{-1}{8\rho} & \dfrac{16\gamma^2-18\gamma+3}{\gamma(8\gamma-5)} \\ \dfrac{\gamma(2\gamma-1)}{8\gamma-3} & \dfrac{8\rho(2\gamma-1)(6\gamma-1)}{8\gamma-5} \end{bmatrix},$$

$\widetilde{\boldsymbol{G}}_1 = \boldsymbol{S}\widetilde{\boldsymbol{G}}_0\boldsymbol{S}, \quad \widetilde{\boldsymbol{G}}_2 = \boldsymbol{S}\widetilde{\boldsymbol{G}}_{-1}\boldsymbol{S}$, and $\rho = \alpha^2\gamma(8\gamma-5)/3$. Two additional parameters δ and τ for the highpass filters can be introduced as described in Theorem 1. Thus the length 4/4 family of SABMFs has three parameters, namely, γ, δ and τ. Through minimizing the deviation of the frequency responses for both matrix lowpass and highpass filters from those of the ideal brickwall filters, we have found a solution with $(\gamma, \delta, \tau) = (0.02491, 0.066063, -0.044016)$. We will refer to the resultant SABMF as the BSA(4/4)* filter. This SABMF has a GMP order of (3,1) but it has the best compression performance among members of this family of length 4/4 SABMFs for a number of image compression experiments we performed.

Appendix 3

A length 5/5 SABMF family

Using Method 2, we obtain a family of length 5/5 SABMFs with GMP order at least (3,3) and approximation order (1,2) with the matrix lowpass sequences given by

$$\boldsymbol{H}_{-1} = \begin{bmatrix} \beta_1 & 2\beta_1 \\ \beta_2 & 2\beta_2 \end{bmatrix}, \quad \boldsymbol{H}_0 = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1-3\gamma}{8\gamma} \\ \dfrac{112\gamma-3}{16(1-16\gamma)} & \dfrac{1}{1-16\gamma} \end{bmatrix}, \quad \boldsymbol{H}_1 = \begin{bmatrix} \dfrac{11\gamma+1}{16\gamma} & 0 \\ 0 & \dfrac{16\gamma+11}{8(16\gamma-1)} \end{bmatrix},$$

$\boldsymbol{H}_2 = \boldsymbol{S}\boldsymbol{H}_0\boldsymbol{S}, \quad \boldsymbol{H}_3 = \boldsymbol{S}\boldsymbol{H}_{-1}\boldsymbol{S}$, and

$$\widetilde{\boldsymbol{H}}_{-1} = \begin{bmatrix} 2\beta_3 & \beta_3 \\ 2\beta_4 & \beta_4 \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}_0 = \begin{bmatrix} \dfrac{1}{2} & 2\gamma \\ \dfrac{1-16\gamma}{16\gamma} & \dfrac{1-16\gamma}{4} \end{bmatrix}, \quad \widetilde{\boldsymbol{H}}_1 = \begin{bmatrix} \dfrac{1}{2}+4\gamma & 0 \\ 0 & \dfrac{128\gamma^2+8\gamma-1}{32\gamma} \end{bmatrix},$$

$\widetilde{\boldsymbol{H}}_2 = \boldsymbol{S}\widetilde{\boldsymbol{H}}_0\boldsymbol{S}, \quad \widetilde{\boldsymbol{H}}_3 = \boldsymbol{S}\widetilde{\boldsymbol{H}}_{-1}\boldsymbol{S}$, with γ being the parameter, and $\beta_1 = \dfrac{5\gamma-1}{32\gamma}$, $\beta_2 = \dfrac{16\gamma-5}{32(1-16\gamma)}$, $\beta_3 = \dfrac{1}{8} - \gamma$ and $\beta_4 = \dfrac{128\gamma^2-24\gamma+1}{64\gamma}$. Note that when $\gamma = 3/14$, this SABMF has a GMP order $(4, 3)$.

The corresponding highpass sequences are obtained by solving the PR conditions, and they are given by

$$\boldsymbol{G}_{-1} = \begin{bmatrix} \dfrac{1}{\rho} & \dfrac{2}{\rho} \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{G}_0 = \begin{bmatrix} \dfrac{\rho-2}{\rho} & 2 \\ \rho-2 & 2\rho \end{bmatrix}, \quad \boldsymbol{G}_1 = \begin{bmatrix} \dfrac{2(1-\rho)}{\rho} & 0 \\ 0 & 4(\rho-1) \end{bmatrix},$$

$\boldsymbol{G}_2 = \boldsymbol{S}\boldsymbol{G}_0\boldsymbol{S}$, $\quad \boldsymbol{G}_3 = \boldsymbol{S}\boldsymbol{G}_{-1}\boldsymbol{S}$, and

$$\widetilde{\boldsymbol{G}}_{-1} = \begin{bmatrix} 2\beta_5 & \beta_5 \\ 2\beta_6 & \beta_6 \end{bmatrix}, \quad \widetilde{\boldsymbol{G}}_0 = \begin{bmatrix} \dfrac{\rho}{4(\rho-2)} & \dfrac{16+11\rho}{64(\rho+2)} \\ \dfrac{6+5\rho}{32\rho(\rho-2)} & \dfrac{1}{8\rho} \end{bmatrix}, \quad \widetilde{\boldsymbol{G}}_1 = \begin{bmatrix} \dfrac{13\rho^2+22\rho-32}{32(4-\rho^2)} & 0 \\ 0 & \dfrac{19\rho-22}{64\rho(\rho-2)} \end{bmatrix},$$

$\widetilde{\boldsymbol{G}}_2 = \boldsymbol{S}\widetilde{\boldsymbol{G}}_0\boldsymbol{S}$, $\quad \widetilde{\boldsymbol{G}}_3 = \boldsymbol{S}\widetilde{\boldsymbol{G}}_{-1}\boldsymbol{S}$, where $\rho = \dfrac{-16\gamma}{8\gamma-1}$, $\beta_5 = \dfrac{3\rho^2+10\rho+32}{128(4-\rho^2)}$ and $\beta_6 = \dfrac{3\rho+10}{128\rho(2-\rho)}$. Introducing the highpass parameters as described in Theorem 1 gives us a set of parameters (γ, τ, δ) for filter design. We have determined through minimizing the closeness to the brick-wall filters that $(\gamma, \tau, \delta) = (0.166, 0.02904, 0.3823)$ which gives us the BSA(5/5)* filter.