A General Approach for Analysis and Application of Discrete Multiwavelet Transforms

Jo Yew Tham, Lixin Shen, Seng Luan Lee, and Hwee Huat Tan

***Abstract*—This paper proposes a general paradigm for the analysis and application of discrete multiwavelet transforms, particularly to image compression. First, we establish the concept of an *equivalent scalar (wavelet) filter bank system* in which we present an equivalent and sufficient representation of a multiwavelet system of multiplicity r in terms of a set of r equivalent scalar filter banks. This relationship motivates a new measure called the *good multifilter properties* (GMP's), which define the desirable filter characteristics of the equivalent scalar filters. We then relate the notion of GMP's directly to the matrix filters as necessary eigenvector properties for the refinement masks of a given multiwavelet system. Second, we propose a generalized, efficient, and nonredundant framework for multiwavelet initialization by designing appropriate preanalysis and post-synthesis multirate filtering techniques. Finally, our simulations verified that both orthogonal and biorthogonal multiwavelets that possess GMP's and employ the proposed initialization technique can perform better than the popular scalar wavelets such as Daubechies'D8 wavelet and the D(9/7) wavelet, and some of these multiwavelets achieved this with lower computational complexity.**

***Index Terms*—Good multifilter properties, image compression, multiwavelets, preanalysis and post-synthesis filtering, wavelets.**

I. Introduction

RECENTLY, much interest has been generated in the study of *multiwavelets* [1]–[4], [6], [12], [15], [21], where *more than one* scaling function and mother wavelet are used to represent a given signal. However, unlike scalar wavelets in which Mallat's pyramid algorithms [10] can be employed directly, the application of multiwavelets requires that the input signal to be first vectorized (which is a problem popularly known as *multiwavelet initialization* or *prefiltering*). To address this problem, Xia *et al.* [21], [22] have proposed new algorithms to compute the initial multiwavelet transform coefficients by using appropriate pre- and post-filtering techniques. Later, Strela *et al.* [15] investigated the construction of "constrained" multiwavelets for filtering two-dimensional (2-D) signals and applied them to image denoising and image compression. Liang *et al.* [9] have also directly applied the GHM multiwavelet [3] to image coding.

In spite of the above research works, there remain a few areas of multiwavelet research that require further investigation for their successful applications, typically in image compression. First, it is observed that there is a lack of "good" (the definition of which will be given in Section III) multiwavelets that will give high energy compaction, which is critical for image compression. Second, there is a need to devise an efficient, robust, and compact representation framework for multiwavelet initialization using any given multiwavelet. Based on the above observations, we are motivated to develop the following solutions:

1) a set of criteria for *analyzing* "good" multiwavelets;
2) a method to *apply* simple but robust and nonredundant preanalysis and post-synthesis multirate filtering for multiwavelet initialization;
3) construction of families of "good" orthogonal and biorthogonal multiwavelets.

In this paper, we will only focus on solutions 1) and 2). Together, they provide a general approach for the design of "good" multiwavelets and the application of *any* multiwavelets, particularly to image compression. The solution (3) is presented in two other papers [12], [17].

The rest of the paper is organized as follows. Section II provides a brief overview of multiwavelets. Section III establishes two proposed concepts: the idea of an equivalent scalar (wavelet) filter bank associated with a multiwavelet system and the notion of "good multifilter properties" (GMP's). Section IV introduces the proposed general framework for multiwavelet initialization. A thorough analysis of the proposals is then carried out in Section V, and the conclusions are drawn in Section VI.

II. Preliminaries of Multiwavelets

For a multiresolution of multiplicity $r > 1$, there are r scaling functions $\phi_1, \ldots, \phi_r$, which are usually written as a vector $\mathbf{\Phi} := (\phi_1, \ldots, \phi_r)^T$ that satisfies the matrix refinement equation (MRE)

$$\mathbf{\Phi}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} \mathbf{H}_k \mathbf{\Phi}(2x - k) \tag{1}$$

where $\mathbf{H}$ is a matrix lowpass filter. The corresponding multiwavelet $\mathbf{\Psi} := (\psi_1, \ldots, \psi_r)^T$ is given by

$$\mathbf{\Psi}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} \mathbf{G}_k \mathbf{\Phi}(2x - k) \tag{2}$$

where $\mathbf{G}$ is a matrix highpass filter. We will also call $\mathbf{H}$ and $\mathbf{G}$ multifilters. Several properties of scaling vectors arising from matrix finite impulse response (FIR) filters, such as orthonormality, stability, smoothness, and good approximation property, have been studied extensively (see, e.g., [2], [6], and [15]).

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The authors are with the Wavelets Strategic Research Programme, Department of Mathematics, National University of Singapore, Singapore (e-mail: thamjy @ wavelets. math. nus. edu. sg; shenlx @wavelets. math. nus. edu. sg; matleesl@wavelets.math.nus.edu.sg; mattanhh@wavelets.math.nus.edu.sg).

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In the Fourier domain, let the *matrix frequency responses* for $\mathbf{H}, \mathbf{G}$ be denoted by $\hat{\mathbf{H}}(\omega), \hat{\mathbf{G}}(\omega)$, respectively, i.e.,

$$\hat{\mathbf{H}}(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \mathbf{H}_k e^{-jk\omega} \text{ and } \hat{\mathbf{G}}(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \mathbf{G}_k e^{-jk\omega} \tag{3}$$

where $j = \sqrt{-1}$. The orthogonality conditions can now be expressed as

$$\hat{\mathbf{H}}(\omega)\hat{\mathbf{H}}^*(\omega) + \hat{\mathbf{H}}(\omega+\pi)\hat{\mathbf{H}}^*(\omega+\pi) = \mathbf{I}_r \tag{4}$$

$$\hat{\mathbf{G}}(\omega)\hat{\mathbf{G}}^*(\omega) + \hat{\mathbf{G}}(\omega+\pi)\hat{\mathbf{G}}^*(\omega+\pi) = \mathbf{I}_r \tag{5}$$

and

$$\hat{\mathbf{H}}(\omega)\hat{\mathbf{G}}^*(\omega) + \hat{\mathbf{H}}(\omega+\pi)\hat{\mathbf{G}}^*(\omega+\pi) = \mathbf{0}_r \tag{6}$$

where * denotes complex conjugate transpose, whereas $\mathbf{I}_r$ and $\mathbf{0}_r$ are the $r \times r$ identity and null matrices, respectively.

III. Analysis of Multiwavelets With Good Filter Characteristics

In this section, we first introduce the concept of an *equivalent scalar (wavelet) filter bank system* to sufficiently represent the multiple-input multiple-output relationship of a given multiwavelet system. We then analyze the good filter properties of the equivalent scalar filter system, which in turn are expressed directly in terms of GMP's of the associated multiwavelet system.

A. Multifilters and Equivalent Scalar Filter Bank System

Consider the multiple-input multiple-ouput (MIMO) relationship of a multifilter system as depicted in Fig. 1(a). In the context of a multiwavelet system with multiplicity $r > 1$, we know that the output vector streams $\mathbf{y}_k, k = 1, 2, \ldots, r$ are given by the convolution of the r input streams $\mathbf{x}_k, k = 1, 2, \ldots, r$ with the $r \times r$ matrix filter impulse response $\mathbf{P}$. Fig. 1(b) illustrates an equivalent and sufficient framework, from an input-output filtering viewpoint, that replaces the multifilter $\mathbf{P}$ with a cascade of a multiplexer, a system of equivalent scalar (wavelet) filters $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r$, and downsamplers. The remaining problems follow: "What is the relationship between the equivalent scalar filters and the associated multifilter system, and what is the multiplexing operator?" In this subsection, we will show that the r equivalent scalar filters are, in fact, the r polyphases of the corresponding multifilter. The second question on multiplexer will be illuminated further in Section IV.

Consider again the MIMO relationship of a multifilter system as portrayed in Fig. 1(a). Let $\mathbf{P}_\ell := (p_{k,m}(\ell))_{k,m=1}^r$, $\ell \in \mathbb{Z}$, $\mathbf{x}_k = \{x_k(n)\}_{n \in \mathbb{Z}}$, and $\mathbf{y}_k = \{y_k(n)\}_{n \in \mathbb{Z}}$. From the definition of matrix filter convolution with vector inputs $\mathbf{x}$, we know that the vector outputs $\mathbf{y}$ are given by

$$y_k(n) = \sum_{\ell \in \mathbb{Z}} \sum_{m=1}^{r} p_{k,m}(\ell) x_m(n-\ell), \quad n \in \mathbb{Z}, \quad k = 1, 2, \ldots, r. \tag{7}$$

Suppose now we multiplex the multiple input streams $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ to produce a single stream $\mathbf{v} = \{v(n)\}_{n \in \mathbb{Z}}$ via a multiple-input-single-output (MISO) operator $\mathbf{MUX}$, as depicted in Fig. 1(b). This allows us to filter the data stream $\mathbf{v}$ independently using each of the r scalar (wavelet) filters $\mathbf{p}_k = \{p_k(n)\}_{n \in \mathbb{Z}}, k = 1, 2, \ldots, r$ to be followed by downsampling with a decimation factor of r, such that

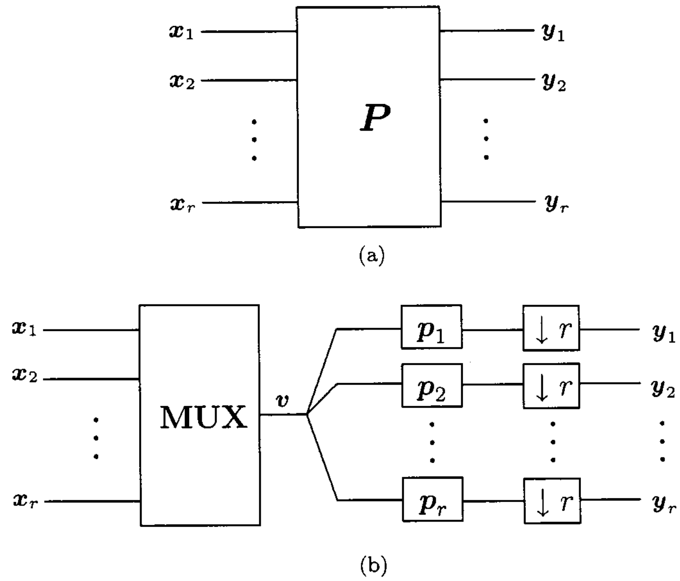

Fig. 1. (a) Multiwavelet filter bank system with multiplicity r. (b) Equivalent r scalar filter bank system with a multiplexer **MUX** and downsamplers.

$$\begin{aligned} y_k(n) &= \sum_{\ell \in \mathbb{Z}} p_k(\ell) v(rn - \ell) \\ &= \sum_{\ell \in \mathbb{Z}} \sum_{m=0}^{r-1} p_k(\ell r + m) v(rn - (\ell r + m)), \quad n \in \mathbb{Z}. \end{aligned} \tag{8}$$

By comparing (7) and (8), the following relationships can be established:

i) The operator $\mathbf{MUX}$, which multiplexes the multiple input streams $\mathbf{x}$ into a scalar data stream $\mathbf{v}$, is defined as

$$\mathbf{MUX}: (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) \mapsto \mathbf{v} : x_m(k) \to v(rk - (m-1)). \tag{9}$$

ii) The equivalent scalar filter bank system $\mathbf{p}_k$ is related to the multifilter $\mathbf{P}_\ell$ by

$$p_k(\ell r + m - 1) = p_{k,m}(\ell), \quad \ell \in \mathbb{Z}, \quad k, m = 1, 2, \ldots, r. \tag{10}$$

Denote $\hat{p}_k(\omega) := \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} p_k(n) e^{-j\omega n}, k = 1, 2, \ldots, r$. From (10), we can show that

$$\hat{p}_s(\omega) = \sum_{t=1}^{r} \left(\sum_{k \in \mathbb{Z}} p_{s,t}(k) e^{-jrk\omega} \right) e^{-j(t-1)\omega} \tag{11}$$

where the tth polyphase of the sth equivalent scalar filter is given by $\sum_{k \in \mathbb{Z}} p_{s,t}(k) \exp[-jrk\omega]$ for all $s, t = 1, 2, \ldots, r$. This clearly shows that we can sufficiently represent any given M-channel multiwavelet system of multiplicity r with M systems of r equivalent scalar filters, each consisting of r polyphases. Typical multiwavelet applications have $M = 2$, and the scalar filters $\mathbf{p}_k$ described here can be either the lowpass $\mathbf{h}_k$ filter or the highpass $\mathbf{g}_k$ filter.

In the above, we have relied on the theory of polyphases of block or vector-valued filter banks [14], [19], [21] to obtain the equivalent representation. It should be noted that Xia [21] also proposed an equivalent representation for multiwavelet systems. The difference between his representation and ours lies in his incorporation of the preprocessing filter in his representation.

From (11), the relationship between the frequency responses of the multifilter and its equivalent scalar filters can be expressed as

$$[\hat{p}_1(\omega), \ldots, \hat{p}_r(\omega)]^T = \hat{\mathbf{P}}(r\omega)\mathbf{e}(\omega) \tag{12}$$

where $\mathbf{e}(\omega) = [1, e^{-j\omega}, \ldots, e^{-j(r-1)\omega}]^T$.

B. Good Multifilter Properties

The main aim of this subsection is to propose a set of new design criteria for multiwavelets called "good multifilter properties" (GMP's). A very interesting property of the relationship that was established earlier is that all the transfer function characteristics of the multiwavelet system can now be analyzed conveniently by designing appropriate equivalent scalar filters. In the following, we will formulate some eigenvector properties of matrix filters corresponding to multiwavelets that possess GMP's by first defining the notion of GMPs for the equivalent scalar filters.

Definition 1: A given multiwavelet system with multiplicity r is considered a good multifilter of GMP order (d_1, d_2, d_3) if its equivalent lowpass and highpass scalar filters $\mathbf{h}_k$ and $\mathbf{g}_k$ possess the following properties:

- $\hat{h}_k^{(\nu)}(0) = \delta_{\nu,0}, \nu = 0, 1, \ldots, d_1 - 1$;
- $\hat{h}_k^{(\nu)}(\pi) = 0, \nu = 0, 1, \ldots, d_2 - 1$;
- $\hat{g}_k^{(\nu)}(0) = 0, \nu = 0, 1, \ldots, d_3 - 1$;

for all $k = 1, 2, \ldots, r$, where the superscript ν denotes the νth derivative, and $d_1 d_2 d_3 \geq 1$.

In fact, the above criteria for good filters follow directly from the fact that $\mathbf{h}_k$ and $\mathbf{g}_k$, $k = 1, 2, \ldots, r$ are the scalar lowpass and highpass filters, respectively. Taking the νth derivative on both sides of (12), the conditions for good filter characteristics can now be expressed explicitly for a multifilter system as

- $$\sum_{q=0}^{\nu} \binom{\nu}{q} r^{\nu-q} \hat{\mathbf{H}}^{(\nu-q)}(0)\mathbf{e}^{(q)}(0) = \delta_{\nu,0}\mathbf{e}(0), \quad \nu = 0, 1, \ldots, d_1 - 1;$$
- $$\sum_{q=0}^{\nu} \binom{\nu}{q} r^{\nu-q} \hat{\mathbf{H}}^{(\nu-q)}(r\pi)\mathbf{e}^{(q)}(\pi) = \mathbf{0}, \quad \nu = 0, 1, \ldots, d_2 - 1;$$
- $$\sum_{q=0}^{\nu} \binom{\nu}{q} r^{\nu-q} \hat{\mathbf{G}}^{(\nu-q)}(0)\mathbf{e}^{(q)}(0) = \mathbf{0}, \quad \nu = 0, 1, \ldots, d_3 - 1. \tag{13}$$

From the above relationship, we can easily verify the following proposition:

Proposition 1: Suppose that an orthogonal multiwavelet system has a GMP order of at least $(1, 1, 1)$; then, we have

i) $\hat{\mathbf{H}}(0)\mathbf{e}(0) = \mathbf{e}(0)$;
ii) $\hat{\mathbf{H}}(r\pi)\mathbf{e}(\pi) = \mathbf{0}$;
iii) $\hat{\mathbf{G}}(0)\mathbf{e}(0) = \mathbf{0}$.

In general, for any orthogonal multiwavelet system with a finite matrix response, $\hat{\mathbf{H}}(0)$ satisfies Condition E and has a vanishing moment of at least order one, i.e., there exists a vector $\mathbf{v}$ such that

$$\mathbf{v}^T\hat{\mathbf{H}}(\pi\nu) = \delta_{\nu,0}\mathbf{v}^T, \quad \nu \in \mathbb{Z}/2\mathbb{Z}. \tag{14}$$

Note that Condition E means all eigenvalues of $\hat{\mathbf{H}}(0)$ are strictly less than one, except for a simple eigenvalue 1 [13]. It also ensures that the solution of the MRE (1) has a unique solution up to a constant multiple.

By setting $\omega = 0$ in the relation (4), left multiplying it with $\mathbf{v}^T$ and applying (14), we have

$$\hat{\mathbf{H}}(0)\mathbf{v} = \mathbf{v}. \tag{15}$$

Similarly, from (6) and (14), we can derive

$$\hat{\mathbf{G}}(0)\mathbf{v} = \mathbf{0}. \tag{16}$$

By combining (1) and the definition of Condition E, we can get

$$\hat{\mathbf{\Phi}}(0) = \mathbf{v} \tag{17}$$

up to a constant (i.e., $\hat{\mathbf{\Phi}}(0)$ is parallel to $\mathbf{v}$). Therefore, if a multiwavelet system has a GMP order of at least $(1, 1, 1)$, then, up to a constant

$$\hat{\mathbf{\Phi}}(0) = \mathbf{e}(0). \tag{18}$$

It is noted that (18) imposes a rather restrictive condition on the design of multiwavelets. However, it is well known that we can perform a change of basis by applying a similarity transformation to the multifilters. The *new* multifilter frequency responses are given by

$$\hat{\mathbf{H}}^{\sim}(\omega) = \mathbf{U}\hat{\mathbf{H}}(\omega)\mathbf{U}^{-1} \quad \text{and} \quad \hat{\mathbf{G}}^{\sim}(\omega) = \mathbf{U}\hat{\mathbf{G}}(\omega)\mathbf{U}^{-1} \tag{19}$$

and the associated *new* multiscaling function vector and multiwavelet vector are defined as

$$\mathbf{\Phi}^{\sim}(x) = \mathbf{U}\mathbf{\Phi}(x) \quad \text{and} \quad \mathbf{\Psi}^{\sim}(x) = \mathbf{U}\mathbf{\Psi}(x) \tag{20}$$

respectively.

Note that such a similarity transformation still guarantees that $\{\mathbf{H}_k^{\sim}, \mathbf{G}_k^{\sim}\}$ satisfies the PR criteria (4)–(6). Therefore, we can say that an orthogonal multiwavelet system $\{\mathbf{H}_k, \mathbf{G}_k\}$ possesses a GMP order (d_1, d_2, d_3) if there exists an orthogonal matrix $\mathbf{U}$ such that $\{\mathbf{H}_k^{\sim}, \mathbf{G}_k^{\sim}\}$ possesses a GMP order (d_1, d_2, d_3). In fact, such an orthogonal matrix $\mathbf{U}$ is completely determined by the moment (zero order) of multiscaling function vector $\hat{\mathbf{\Phi}}(0)$ such that $\mathbf{U}\hat{\mathbf{\Phi}}(0)$ is parallel to vector $\mathbf{e}(0)$.

Finally, for $r = 2$, we have the following proposition.

Proposition 2: A given orthogonal multiwavelet system of multiplicity $r = 2$ has a GMP order of at least $(1, 1, 1)$ iff $\hat{\mathbf{H}}(0)$ is singular.

Proof—Sufficient Part: Choose an orthogonal matrix $\mathbf{U}$ such that the vector $\mathbf{U}\hat{\mathbf{\Phi}}(0)$ is parallel to $\mathbf{e}(0) = [1, 1]^T$. Clearly, from (15) and (16), we know that the GMP order components d_1 and d_3 are at least 1. In addition, suppose that $\hat{\mathbf{H}}^{\sim}(0) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$; then, from (14) and (15), we have $a = d, b = c$. On the other hand, since $\hat{\mathbf{H}}(0)$ is singular and

$$\hat{\mathbf{H}}^{\sim}(0)\mathbf{e}(0) = \mathbf{e}(0) \tag{21}$$

we have $a = b = c = d = 1/2$, which implies that $\hat{\mathbf{H}}^{\sim}(0)\mathbf{e}(\pi) = \mathbf{0}$. Hence, $d_2 \geq 1$. The proof for the necessary part is obvious from Proposition 1. □

It is noteworthy that a similar concept of "balanced multiwavelets" were introduced in [8], where the concern was to achieve "good balance" of the r scaling functions. This amounts to imposing condition i) in Proposition 1. Our concept of GMP's goes further in that we further impose condition ii) in Proposition 1 with the aim of preventing "DC leakage" and the checkerboard artifacts in the reconstructed images [14, p. 368]. Some results verifying the importance of this design criterion will be shown later.

IV. Application of Discrete Multiwavelet Transforms

The section focuses on how to efficiently and effectively apply discrete multiwavelet decomposition and reconstruction to a given signal. In particular, we attempt to address the important issue of *multiwavelet initialization or prefiltering*, which concerns the generation of multiple (vector) input streams from a given scalar source stream. Several proposals (e.g., [21] and [22]) for prefiltering have preceded this paper. Our approach, however, is significantly different. We propose to develop a generalized paradigm for discrete multiwavelet transforms, which works well with any given multiwavelet system, regardless of whether it possesses GMP's or not. In addition, the proposed multiresolution framework also embodies the following properties: orthogonality, low complexity, and compact (or nonredundant) representation of the input signal. For simplicity of exposition, but without loss of generality, we consider only orthogonal multiwavelet systems with multiplicity $r = 2$ for the rest of the paper, unless mentioned otherwise.

A. Proposed Preanalysis and Post-Synthesis Multirate Filtering

A multifilter system needs to operate on vector input streams. The problem boils down to how we can obtain the vector inputs $\mathbf{s}_{0,k} = [s_{1,0,k}, s_{2,0,k}]^T$ from a given scalar input signal $\mathbf{f} = \{f_k\}$. For the case of multiplicity $r = 2$, we essentially pair up the adjacent data of the scalar input stream to generate the vector input stream $\mathbf{s}_{0,k} = \int f(x)\phi(2x-k)\,dx \approx [f_{2k}, f_{2k+1}]^T$. This is analogous to what happens in the scalar case where the common and popular choice for approximating $s_{0,k} = \int f(x)\phi(2x-k)\,dx$ is the one-point quadrature [16], where $s_{0,k} \approx f(k)$. Here, we assume that elements of $\mathbf{f}$ are locally smooth, and hence, $f_{2k} \approx f_{2k+1}$ with $\mathbf{s}_{0,k} \approx \sigma(1,1)^T$ for some real constant σ. Using (19), (21), and the assumption on $\mathbf{s}_{0,k}$, we have $\mathbf{U}\hat{\mathbf{H}}(0)\mathbf{U}^{-1}\mathbf{s}_{0,k} \approx \mathbf{s}_{0,k}$. This allows us to derive the preanalysis and post-synthesis operators as follows:

i) Preanalysis operator (**PRE**):

$$\mathbf{PRE}: \mathbf{v} \rightarrow \mathbf{U}^{-1}\mathbf{v}$$

ii) Post-synthesis operator (**POST**):

$$\mathbf{POST}: \mathbf{v} \rightarrow \mathbf{U}\mathbf{v}$$

where the orthogonal matrix $\mathbf{U}$ has two possible forms:

$$\mathbf{U} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \mathbf{U} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \tag{22}$$

where $\theta = -\pi/4$ when $\hat{\phi}_1(0) = 0$, or, otherwise, $\theta = \pi/4 - \tan^{-1}(\hat{\phi}_2(0)/\hat{\phi}_1(0))$ for $\theta \in (-\pi/4, \pi/4]$.

A natural question now arises as to which orthogonal matrix $\mathbf{U}$ should be used for a given multiwavelet system. In order to help us determine the better prefilter, we have investigated a number of other filter design criteria and concluded that the following measure of deviation from the ideal "brickwall" lowpass filter is both reliable and consistent.

$$E = \sum_{\nu=1}^{2} \int_0^{\frac{\pi}{2}} (\sqrt{2} - |\widehat{h_\nu^{\sim}}(\omega)|)^2 d\omega + \int_{\frac{\pi}{2}}^{\pi} |\widehat{h_\nu^{\sim}}(\omega)|^2 d\omega$$

where $\widehat{h_1^{\sim}}(\omega)$ and $\widehat{h_2^{\sim}}(\omega)$ are the equivalent scalar lowpass filters associated with the transformed multiwavelets. We select the matrix $\mathbf{U}$ that gives the smaller value of E.

B. Proposed Generalized Framework for Discrete Multiwavelet Transforms

With prefiltering, we are now able to generate the multiple input streams $\mathbf{s}_{0,k}, k \in \mathbb{Z}$, which form the initial expansion coefficients of a given multiwavelet system. For multiscale analysis of a signal, we can employ Mallat's multiresolution algorithm [10], which recursively decomposes and critically downsamples the smooth (approximation) version of the signal at each scale. The *multiwavelet decomposition algorithm* is given as

$$\begin{aligned} \mathbf{s}_{m-1,k} &= \sum_{\ell \in \mathbb{Z}} \mathbf{H}_{\ell-2k}\mathbf{s}_{m,\ell} \quad \text{and} \\ \mathbf{t}_{m-1,k} &= \sum_{\ell \in \mathbb{Z}} \mathbf{G}_{\ell-2k}\mathbf{s}_{m,\ell}, \quad m,k \in \mathbb{Z}. \end{aligned} \tag{23}$$

Essentially, we are performing the lowpass and highpass filtering with the matrix QMF's $\mathbf{H}$ and $\mathbf{G}$, respectively. The synthesis stage, which recombines the approximation and detail information of the signal, can then be carried out using the *multiwavelet reconstruction algorithm*

$$\mathbf{s}_{m,k} = \sum_{\ell \in \mathbb{Z}} \mathbf{H}_{k-2\ell}^T \mathbf{s}_{m-1,\ell} + \sum_{l \in \mathbb{Z}} \mathbf{G}_{k-2\ell}^T \mathbf{t}_{m-1,\ell}, \quad m,k \in \mathbb{Z}. \tag{24}$$

Fig. 2 illustrates the proposed generalized framework for discrete multiwavelet decomposition and reconstruction of a 2-D signal using separable transformation along each dimension. The integration of the proposed multirate preanalysis and post-synthesis filters with Mallat's multiscale algorithms is also shown. Clearly, the overall framework provides a compact

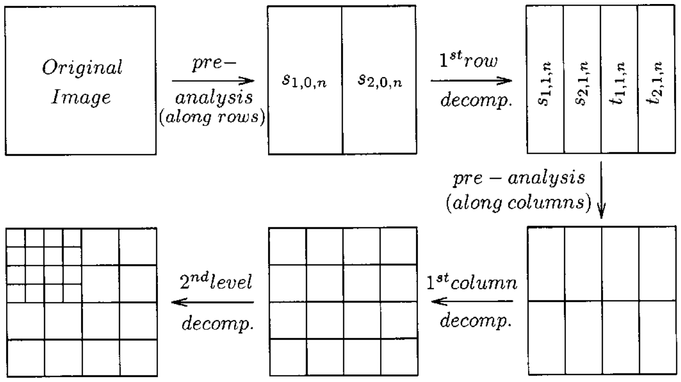

Fig. 2. Integration of the proposed multirate preanalysis (post-synthesis) filtering for nonredundant discrete multiwavelet decomposition of a 2-D image. The discrete multiwavelet reconstruction is obtained by reversing the above process and with the preanalysis processes replaced by the post-synthesis processes.

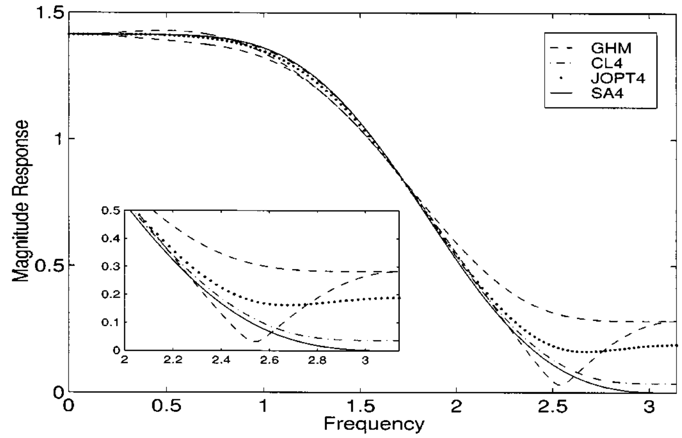

Fig. 3. Magnitude responses of the equivalent scalar filters associated with orthogonal multiwavelets GHM, CL4, JOPT4, and SA4.

(nonredundant) representation of the original signal, which is critical for image compression. Note that the row-wise (column-wise) preanalysis filtering is only applied before the *first* level of row (column) decomposition. In general, an N-level multiwavelet decomposition of a 2-D image will produce $4(3N+1)$ subbands. Such a subband structure closely resembles that of a full wavelet-packet decomposition using scalar wavelets. In spite of the similarity in their subband structures, we will show later that the image compression performance using multiwavelets is more superior to that using scalar wavelet packets.

V. Performance Analysis and Comparison

This section investigates our proposals by comparing them with some existing methods and filters. In the following simulations, we will only concentrate on applying the proposed techniques to image compression, although we believe that such a framework should also produce encouraging results in other applications such as image denoising and enhancement. We will first analyze the effectiveness, efficiency, and robustness of the proposed prefiltering technique. Image compression results using various prefilters are compared. This is followed by a comparison of image compression performances using different multiwavelets and scalar wavelets. The application of the proposed generalized framework for multiwavelets and the use of wavelet-packet for scalar wavelets are investigated. For fair comparisons, the same still image codec (SPIHT) by Said and Pearlman [11] is used[1] throughout for compressing different images at various bit rates. Both the objective measure of peak signal-to-noise ratio (PSNR) and subjective evaluation of the quality of reconstructed images are presented.

A. Examples of Multiwavelets

In our image compression experiments, both orthogonal and biorthogonal scalar wavelets and multiwavelets were used for comparison. For the orthogonal case, the following four 4-tap orthogonal multiwavelet filters are chosen for their respective properties:

i) GHM multiwavelet [3] has symmetic scaling functions and an approximation order of 2.
ii) Chui and Lian's (CL4) multiwavelet [2] has the highest possible approximation order of 3 for its filter length.
iii) Jiang's multiwavelet (JOPT4) [7] has optimal time-frequency localization for its filter length.
iv) Our proposed multiwavelet (SA4) [12] has an approximation order of 1 and a GMP order of $(1,2,1)$.

The CL4, JOPT4, and SA4 multiwavelets belong to a class of multifilters which scaling and wavelet functions are symmetric/antisymmetric pairs [2]. For 4-tap multiwavelets with GMP order of at least $(1,1,1)$, the lowpass filter matrices $\mathbf{H}_k$, $\mathbf{G}_k$, $k=0,1,2,3$ satisfy

$$\mathbf{H}_0=\frac{\sqrt{2}}{2}\begin{bmatrix}\frac{1}{\alpha^2+1} & \frac{\alpha}{\alpha^2+1}\\ \frac{1}{\alpha^2+1} & \frac{-\alpha}{\alpha^2+1}\end{bmatrix},\quad \mathbf{H}_1=\frac{\sqrt{2}}{2}\begin{bmatrix}\frac{\alpha^2}{\alpha^2+1} & \frac{\alpha}{\alpha^2+1}\\ \frac{-\alpha^2}{\alpha^2+1} & \frac{\alpha}{\alpha^2+1}\end{bmatrix}$$

$$\mathbf{G}_0=\frac{\sqrt{2}}{2}\begin{bmatrix}\frac{\alpha}{\alpha^2+1} & \frac{-1}{\alpha^2+1}\\ \frac{\alpha}{\alpha^2+1} & \frac{1}{\alpha^2+1}\end{bmatrix},\quad \mathbf{G}_1=\frac{\sqrt{2}}{2}\begin{bmatrix}\frac{-\alpha}{\alpha^2+1} & \frac{\alpha^2}{\alpha^2+1}\\ \frac{\alpha}{\alpha^2+1} & \frac{\alpha^2}{\alpha^2+1}\end{bmatrix}$$

$\mathbf{H}_k=\mathbf{S}\mathbf{H}_{3-k}\mathbf{S}$, $\mathbf{G}_k=\mathbf{S}\mathbf{G}_{3-k}\mathbf{S}$ for $k=2,3$, where $\mathbf{S}=\mathrm{diag}[1,-1]$. For example, the SA4 multiwavelet is obtained when $\alpha=4+\sqrt{15}$. The magnitude responses of the equivalent scalar filters associated with each of the above multiwavelets are plotted in Fig. 3. It is also clear from the plot that only SA4 possesses GMP's, which are manifested in the magnitude responses as smooth decays to zero at $\omega=\pi$.

For the biorthogonal case, we use the class of symmetric-antisymmetric biorthogonal multiwavelets (SABMW's) constructed in [17] to have GMP's. In particular, we will employ the length 4/4 BSA(4/4) filter, the length 6/6 BSA(6/6) filter, and the length 7/9 BSA(7/9) filter for comparing with the popular scalar filters D(7/9) and V(10/18) [20].

B. Performance Comparison of Prefiltering Techniques

The main aim of this subsection is to compare and contrast the proposed prefilter against two other existing prefilters. The prefilters involved are

i) Hardin and Roach's prefilter (HRP) [5], which is orthogonal and approximation-order preserving;
ii) the Xia *et al.* prefilter (XP) [21], which is interpolatory;

[1] Similar relative performance comparisons were also observed using other image codecs such as [18].

TABLE I
COMPARISONS OF PSNR VALUES (IN DECIBELS) OF DIFFERENT IMAGES AT DIFFERENT COMPRESSION RATIOS (CR) USING THE GHM MULTIWAVELET BUT WITH THREE DIFFERENT PRE-FILTERING METHODS

Image	CR	HRP	XIP	TP
Lena	8:1	40.22	39.61	40.40
	16:1	36.82	36.10	36.89
	32:1	33.61	32.79	33.58
	64:1	30.50	30.11	30.52
	128:1	27.76	27.63	27.75
Barbara	8:1	36.10	34.20	36.31
	16:1	31.11	29.54	31.14
	32:1	27.45	26.47	27.44
	64:1	24.83	24.54	24.85
	128:1	23.47	23.26	23.43
Boat	8:1	37.41	37.11	38.35
	16:1	33.34	32.73	33.65
	32:1	30.00	29.53	30.04
	64:1	27.37	27.04	27.51
	128:1	25.27	25.05	25.25
Goldhill	8:1	35.80	34.99	36.02
	16:1	32.44	31.85	32.51
	32:1	29.87	29.46	29.87
	64:1	27.83	27.55	27.83
	128:1	25.96	25.76	25.93

TABLE II
COMPARISONS OF PSNR VALUES (IN DECIBELS) USING DIFFERENT IMAGES AT DIFFERENT COMPRESSION RATIOS (CR) FOR THE ORTHOGONAL MULTIWAVELET FILTERS: i) GHM; ii) CL4; iii) JOPT4; iv) SA4 MULTIWAVELET WITH GMP ORDER $(1, 2, 1)$. THE SCALAR FILTER D8 AND A SPECIAL WAVELET-PACKET IMPLEMENTATION OF D8, DENOTED BY DP8 ARE INCLUDED HERE FOR BENCH-MARKING PURPOSES

	CR	GHM	CL4	JOPT4	SA4	D8	DP8
Lena	32:1	33.58	34.37	34.01	34.42	34.00	33.64
	64:1	30.52	31.40	31.14	31.46	30.97	30.64
	128:1	27.75	28.73	28.54	28.75	28.33	28.01
Barbara	16:1	31.14	32.06	31.64	32.13	31.67	31.99
	32:1	27.44	28.23	28.07	28.30	27.86	28.18
	64:1	24.85	25.53	25.47	25.56	25.12	25.41
Boat	16:1	33.65	34.46	34.09	34.49	33.95	33.62
	32:1	30.04	30.85	30.59	30.87	30.44	30.15
	64:1	27.37	28.08	27.85	28.10	27.74	27.54
Goldhill	16:1	32.51	33.12	32.96	33.14	32.66	32.52
	32:1	29.87	30.58	30.45	30.60	30.09	29.90
	64:1	27.83	28.50	28.41	28.52	28.00	27.92

iii) our proposed prefilter (TP), which uses a simple orthogonal transformation.

Since the objective is to compare the significance of different prefilters, we have adopted the popular GHM multiwavelet as the common[2] multifilter in our simulations. However, it should be emphasized that the proposed TP framework is robust enough to work well with any given multiwavelet.

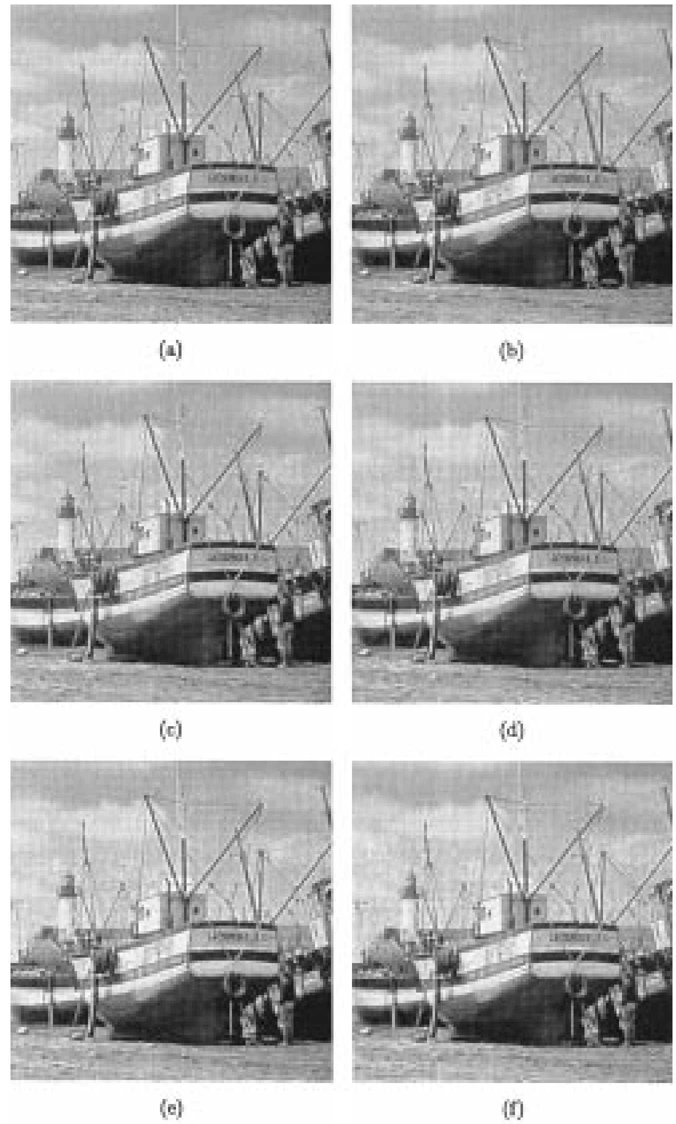

Fig. 4. (a) Original Boat image. Reconstructed images at compression ratio 32:1 using (b) D8 (30.44 dB), (c) GHM (30.04 dB), (d) CL4 (30.85 dB), (e) JOPT4 (30.59 dB), and (f) SA4 (30.87 dB).

Table I illustrates the PSNR results of the three prefiltering schemes over a wide range of compression ratios. It can be observed that the TP and HRP methods have comparable performance, whereas both methods consistently perform better than the XP method. However, it should be pointed out that the TP method has lower computational complexity than the HRP method. For the case of GHM multiwavelet, the TP method needs only one matrix-vector multiplication as compared with the HRP method, which requires two such multiplications. Furthermore, for the class of symmetric-antisymmetric multiwavelets $\hat{\phi}_1(0) = 0$ and from (22), $\mathbf{U} = 1/\sqrt{2}\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$. Here, if the normalization constant of $1/\sqrt{2}$ can be absorbed into the filters for the first level of decomposition and the first level of reconstruction, then the TP method requires practically no multiplication and only two additions for each input vector.

[2]The choice of GHM multiwavelet is also motivated by the existing prefilters whose designs are based on GHM.

TABLE III
IMAGE COMPRESSION PERFORMANCE COMPARISONS OF POPULAR BIORTHOGONAL SCALAR WAVELET FILTERS [D(7/9), V(10/180], AND THREE SYMMETRIC-ANTISYMMETRIC BIORTHOGONAL MULTIFILTERS BSA (4/4), BSA (6/6), AND BSA (7/9) CONSTRUCTED IN [17] WITH GOOD MULTIFILTER PROPERTIES

Filter	**Lena** Compression ratio					**Barbara** Compression ratio				
	8:1	16:1	32:1	64:1	128:1	8:1	16:1	32:1	64:1	128:1
D(7/9)	41.01	37.83	34.74	31.75	29.04	37.45	32.10	28.13	25.38	23.77
V(10/18)	41.08	37.93	34.83	31.86	29.08	38.02	32.50	28.32	25.30	23.78
BSA(4/4)	41.01	37.84	34.77	31.83	29.06	37.71	32.46	28.56	25.57	24.04
BSA(6/6)	41.11	37.97	34.96	31.94	29.13	37.99	32.68	28.69	26.01	23.98
BSA(7/9)	41.18	38.04	34.99	31.96	29.10	38.10	32.85	28.73	25.90	24.02

C. Performance Comparison of Scalar Wavelet and Multiwavelet Filters

This subsection aims to demonstrate the importance of GMP's as a set of new design criteria for constructing multiwavelets, particularly for image compression. We first compare in the orthogonal settings where the four multiwavelet filters described in Section V-A were compared. For fair comparisons and to illustrate the robustness of the proposed framework, we used the TP technique as the common prefilter for all multiwavelets. We have also included both Daubechies' 8-tap (D8) scalar wavelets for benchmarking purposes.

Table II compares the PSNR performances of different orthogonal wavelet filters for a number of images and bit rates. Among the three length-4 symmetric-antisymmetric orthogonal multiwavelets, it is clear that SA4 generally outperforms both JOPT4 and CL4. It is also worth noting that the performance of CL4 is very close to that of SA4. Such results are expected as the lowpass magnitude responses of CL4 are close to zero (about 0.051) at $\omega = \pi$ (see Fig. 3). That is, CL4 has "near" GMP's. In addition, SA4 can outperform scalar wavelets D8 by more than 0.54 dB. These results are very encouraging as the computational complexity of SA4 is only half of that of D8. The subband structure of SA4 resembles that of a special case of wavelet-packet decomposition using scalar wavelets. For fair and comprehensive comparisons, we also applied D8 using wavelet-packet decomposition with the same subband structure. The results are shown under the heading "DP8" in Table II. While DP8 performs better than D8 for "busy" images such as Barbara, its performance is still worse than that of SA4.

Fig. 4 depicts the subjective reconstructed image quality of the standard Boat image at a compression of 32:1 using different orthogonal wavelet filters. It is observed that the image quality corresponds well with the objective measure of PSNR. A careful comparison, however, reveals some distinct differences among the reconstructed images. For example, the white mast at the top center of the image is clearly missing or severely distorted in the images, except for those reconstructed using CL4 and SA4.

It is also noteworthy that SA4 can achieve very competitive compression performances even though it has lower approximation order than both GHM and CL4 and worse time-frequency localization than JOPT4. This reflects the significance of GMP's as a useful set of design criteria for the construction of multiwavelets targeting image compression applications.

While SA4 performs well against other length 4 orthogonal multiwavelets and D8, it performs worse than the popular biorthogonal scalar filter D(9/7). To address this problem, we generalized GMP's to the biorthogonal multiwavelet case in [17] and constructed symmetric-anitisymmetric biorthogonal multiwavelets with GMP's and use them for comparisons against D(9/7). Table III shows the image compression performance comparisons of popular biorthogonal scalar filters D(7/9) and V(10/18) [20] and three SABMW filters BSA(4/4), BSA(6/6), and BSA(7/9). It should be noted that BSA(4/4) outperforms D(9/7), requiring only two thirds of the computational complexity for D(7/9) [17], whereas BSA(6/6) outperforms V(10/18) with 11/14 of its computatonal complexity.

VI. CONCLUSION

A generalized paradigm for the analysis and application of discrete multiwavelets to image compression was presented in this paper. We introduced the idea of an *equivalent scalar (wavelet) filter bank system*, which provides a sufficient representation of the multiple-input multiple-output relationship of a given multiwavelet system. We showed that the r equivalent scalar filters are, in fact, the r polyphases of the corresponding multifilter, and the multiplexer operation actually motivated the development of the proposed pre-filtering framework. The notion of *good multifilter properties* (GMP's) was then proposed as a new tool for analysis, construction, and application of good multiwavelets for image compression. We also presented the necessary and sufficient conditions for determining the existence of GMP's, explained the eigenvector properties of the matrix refinement mask associated with multiwavelets possessing GMP's, and defined the GMP order of a given multiwavelet system. Next, we proposed a generalized preanalysis and post-synthesis framework for multiwavelet initialization and showed how they can produce an integrated solution for multiresolution image analysis. The proposed framework is not only robust to work well with any given discrete multiwavelets, but it is also orthogonal, low in complexity, and provides a compact (nonredundant) representation of the input signal. Finally, extensive simulations in image compression verified the significance of GMP's and the efficiency of the proposed framework for discrete multiwavelet transforms. More details about our work can be obtained via the Web at http://wavelets.math.nus.edu.sg/projects/multiwavelets/.

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Jo Yew Tham received the B.Eng. (Hon.) degree in electrical engineering from the National University of Singapore (NUS) in 1995. Currently, he is completing the Ph.D. degree in electrical engineering at NUS.

He is working as an R&D Manager in San Jose, CA, on a server-based video messaging system over the Internet. His current active research interests include highly scalable (wavelet-based) image/video compression algorithms, design and application of multiwavelet filterbanks, scalable motion estimation, layered joint source-channel coding, packetization and error-control coding, and Internet-based multimedia applications.

Mr. Tham received the 1995 NUS Engineering Innovation Award and the 1995 IEEE (Region 10 UG Category) Best Student Paper Award for his work on detail preserving image compression using wavelets transform. He is a member of the Malaysian MENSA Society.

Lixin Shen received the B.Sc. and M.Sc. degrees from Peking University, Beijing, China, in 1987 and 1990, respectively, and the Ph.D. degree from Zhongshan University, Guangzhou, China, in 1996, all in mathematics.

He is now a Research Fellow at Centre for Wavelets, Approximation, and Information Processing (CWAIP), National University of Singapore. His current research interests are filterbanks and wavelets with application to image processing.

Seng Luan Lee received the B.Sc. (Hons.) and M.Sc. degrees from the University of Malaya, Kuala Lumpur, in 1969 and 1971, respectively, and the Ph.D. degree in mathematics from the University of Alberta, Edmonton, Alta., Canada, in 1974. He is a Professor of mathematics and Director of the Centre for Wavelets, Approximation, and Information Processing (CWAIP) at the National University of Singapore. His research interests include spline approximation, computer-aided geometric design, and wavelets.

Hwee Huat Tan received the B.Sc. (Hons.) and Ph.D. degrees from the University of Adelaide, Adelaide, Australia, in 1987 and 1989, respectively, both in applied mathematics.

He is currently a Senior Lecturer at the Department of Mathematics, National University of Singapore. His current research interests includes theory and application of wavelets.