Good Multifilter Properties: A New Tool For Understanding Multiwavelets

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Abstract *This paper introduces a new tool called "good multifilter properties" (GMPs) to help us better understand the characteristics of a useful multiwavelet (multifilter) system, with particular emphasis on the application to image compression. We first formulate the concept of an equivalent system of scalar filters, in which we derive an equivalent and sufficient representation of a given multiwavelet system with multiplicity r in terms of a set of r scalar filter banks. This relationship leads the notion of GMPs which defines the desirable filter responses of the equivalent scalar filters. It is then related back to the matrix filters as necessary eigenvector properties for the refinement masks of a given multiwavelet system. We further show how the above ideas, when combined with a suitable similarity transformation of the matrix filters, can lead to an efficient and general framework for multiwavelet initialization. Finally, our simulation results verified that both orthogonal and biorthogonal multiwavelets possessing GMPs, and using the proposed pre-filtering technique, can give significant improvements in image compression performance and lower computational requirement when compared with some popular scalar wavelets.*

Keywords: wavelets, multiwavelets, pre-filtering, compression, good multifilter properties.

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1 Introduction

The recent spate of research activities in multiwavelets provides a good indication of the importance and potential impact of multiwavelets on signal processing. In spite of the extensive theoretical research and successful application of scalar wavelets, there remain gaps that need to be bridged before multiwavelets can be used both efficiently and effectively. It is also well-known that multiwavelets can *simultaneously* possess orthogonality, linear phase symmetry, and a shorter support for a given number of vanishing moments [2], and multiwavelets have great potentials for applications such as image compression and denoising. Having realized this, it becomes critical to study and better understand what factors will contribute to good multiwavelets and their applications. The main object of this paper is to introduce a new analytical tool called *good multifilter properties* (GMPs) for both multiwavelet filter design and application.

A vector function $\mathbf{\Phi} := (\phi_1, \ldots, \phi_r)^T$, where r is a fixed positive integer and $\phi_j, j = 1, 2, \ldots, r$, are compactly supported functions in $L^2(\mathbb{R})$, is called an *orthogonal multiscaling function* if it generates an *orthogonal multiresolution analysis with multiplicity* r [3], such that it satisfies the following refinement equation:

$$\mathbf{\Phi}(x) = \sum_{k\in\mathbb{Z}} \boldsymbol{H}_k \mathbf{\Phi}(2x - k), \qquad (1)$$

where $\boldsymbol{H}_k$ is a finitely supported sequence of $r \times r$ matrices. The corresponding multiwavelet function vector $\mathbf{\Psi} = (\psi_1, \ldots, \psi_r)^T$ satisfies

$$\mathbf{\Psi}(x) = \sum_{k\in\mathbb{Z}} \boldsymbol{G}_k \mathbf{\Phi}(2x-k), \qquad (2)$$

for some finitely supported sequence of $r \times r$ matrices $\boldsymbol{G}_k$.

In the Fourier domain, the two-scale refinement equations (1) and (2) can be written as

$$\begin{aligned} \widehat{\mathbf{\Phi}}(2\omega) &= \widehat{\boldsymbol{H}}(\omega)\widehat{\mathbf{\Phi}}(\omega), & (3) \\ \widehat{\mathbf{\Psi}}(2\omega) &= \widehat{\boldsymbol{G}}(\omega)\widehat{\mathbf{\Phi}}(\omega), & (4) \end{aligned}$$

where $\widehat{\boldsymbol{H}}(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{H}_k e^{-ik\omega}$ and $\widehat{\boldsymbol{G}}(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{G}_k e^{-ik\omega}$ are the *matrix lowpass* and *matrix highpass frequency responses*, respectively. In order to ensure perfect reconstruction (PR), the matrix filters must also satisfy the following relations:

$$\begin{aligned} \widehat{\boldsymbol{H}}(\omega)\widehat{\boldsymbol{H}}^*(\omega) + \widehat{\boldsymbol{H}}(\omega+\pi)\widehat{\boldsymbol{H}}^*(\omega+\pi) &= \boldsymbol{I}, & (5) \\ \widehat{\boldsymbol{H}}(\omega)\widehat{\boldsymbol{G}}^*(\omega) + \widehat{\boldsymbol{H}}(\omega+\pi)\widehat{\boldsymbol{G}}^*(\omega+\pi) &= \mathbf{0}, & (6) \\ \widehat{\boldsymbol{G}}(\omega)\widehat{\boldsymbol{G}}^*(\omega) + \widehat{\boldsymbol{G}}(\omega+\pi)\widehat{\boldsymbol{G}}^*(\omega+\pi) &= \mathbf{0}, & (7) \end{aligned}$$

where the superscript * denotes conjugate transpose. Specifically, a matrix sequence $\{\boldsymbol{H}_k\}$ that satisfies (5) is called a conjugate quadrature filter (CQF).

The rest of the paper is organized as follows. Sec. 2 first establishes the relationship between a given multiwavelet system and its *equivalent system of scalar filters*. It then introduces the notion of GMPs, both in the scalar and multifilter frameworks. Sec. 3 proposes an efficient pre-filtering technique which works well with multifilters possessing GMPs. Finally Sec. 4 shows that the proposed ideas can lead to significant improvement in image compression performance.

2 Equivalent Scalar Filters and GMPs

In order to better understand the multiple-input multiple-output (MIMO) relationship of a multifilter system, we first formulate the concept of an *equivalent system of scalar filters* to represent a given multifilter in terms of a set of equivalent scalar filter banks. For a given multifilter $\boldsymbol{P}$ as depicted in Fig. 1 (a), the following proposition will illuminate the above concept via a multiplexing operation:

Proposition 1 *Consider a multiwavelet system with multiplicity $r > 1$ that has r input streams, $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_r$, and r output streams, $\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_r$. Then, there always exists an equivalent filter bank system with a set of r scalar (wavelet) filters, $\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_r$, such that the output stream $\boldsymbol{y}_k$ is a filtered version of a multiplexed input stream, $\boldsymbol{v}$, with the scalar filter $\boldsymbol{p}_k$, for all $k = 1, 2, ..., r$.*

The MIMO relation of a multiwavelet filter bank system with multiplicity r can be represented as the convolution of the r input streams, $\boldsymbol{x}_k, k = 1, 2, ..., r$, with the $r \times r$ matrix filter impulse response $\boldsymbol{P}$. In the Fourier domain, it can be written as

$$\widehat{\boldsymbol{Y}}(\omega) = \widehat{\boldsymbol{P}}(\omega)\widehat{\boldsymbol{X}}(\omega), \qquad (8)$$

where $\widehat{\boldsymbol{P}}(\omega) = \sum_{\ell\in\mathbb{Z}} \boldsymbol{P}_\ell e^{-j\ell\omega}$ is the filter's frequency response, $\widehat{\boldsymbol{X}}(\omega) = (\widehat{\boldsymbol{x}}_1(\omega), ..., \widehat{\boldsymbol{x}}_r(\omega))^T$, and $\widehat{\boldsymbol{Y}}(\omega) = (\widehat{\boldsymbol{y}}_1(\omega), ..., \widehat{\boldsymbol{y}}_r(\omega))^T$. Let $\boldsymbol{P}_\ell := (p_{i,j}(\ell))_{i,j=1}^r$, $\ell \in \mathbb{Z}$, $\boldsymbol{x}_k = \{x_k(n)\}_{n\in\mathbb{Z}}$, and $\boldsymbol{y}_k = \{y_k(n)\}_{n\in\mathbb{Z}}$. For the multiwavelet system shown in Fig. 1 (a), we have

$$y_k(n) = \sum_{\ell\in\mathbb{Z}} \sum_{j=1}^{r} p_{k,j}(\ell) x_j(n-\ell), \quad n \in \mathbb{Z}, \qquad (9)$$

for all $k = 1, 2, ..., r$. In the meantime, assume that we already have the multiple (vector) input streams $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_r$, which are filtered to produce the vector output streams $\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_r$. An efficient technique for generating the vector input streams will be given later in Sec. 3. Fig. 1 (b) shows the equivalent and sufficient framework, from an input-output filtering viewpoint, that replaces the multifilter $\boldsymbol{P}$ with a cascade of a multiplexer, a system of equivalent scalar filters $\boldsymbol{p}_1, ..., \boldsymbol{p}_r$, and downsamplers.

The object now is to determine the multiple-input-single-output (MISO) multiplexer operator, **MUX**, as well as the corresponding relation of the set of equivalent scalar filters

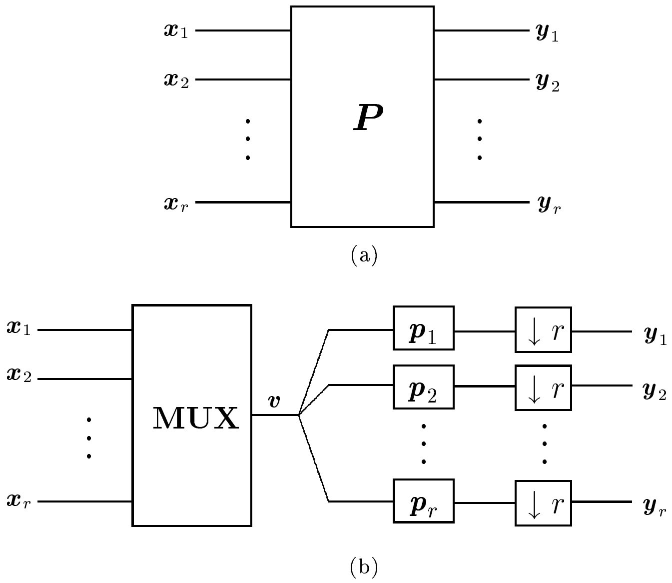

Figure 1: Illustration of the concept of an equivalent system of scalar filters: (a) Multifilter framework, and (b) Equivalent scalar filter framework with a multiplexer and downsamplers.

with the matrix filter $\boldsymbol{P}$. Suppose that we now filter the single multiplexed stream $\boldsymbol{v} = \{v(n)\}_{n\in\mathbb{Z}}$ independently using each of the r scalar (wavelet) filters, $\boldsymbol{p}_k = \{p_k(n)\}_{n\in\mathbb{Z}}, k = 1, 2, ..., r$, such that:

$$\begin{aligned} y_k(n) &= \sum_{\ell\in\mathbb{Z}} p_k(\ell)v(rn-\ell), \qquad (10) \\ &= \sum_{\ell\in\mathbb{Z}}\sum_{j=0}^{r-1} p_k(\ell r+j)v(rn-(\ell r+j)), \end{aligned}$$

for $n \in \mathbb{Z}$, and the scalar filter impulse responses are given by some combinations of the set $\{p_{k,j}(\ell)\}, \ell \in \mathbb{Z}, j, k = 1, 2, ..., r$.

In most signal processing applications, the matrix filter $\boldsymbol{P}$ is usually either a lowpass filter $\boldsymbol{H}$, or a highpass filter $\boldsymbol{G}$. Let $\boldsymbol{H}_\ell = (h_{k,j}(\ell))_{k,j=1}^r$ and $\boldsymbol{G}_\ell = (g_{k,j}(\ell))_{k,j=1}^r$. It can be shown [7] that the equivalent scalar filters $\boldsymbol{h}_k = \{h_k(n)\}_{n\in\mathbb{Z}}$ and $\boldsymbol{g}_k = \{g_k(n)\}_{n\in\mathbb{Z}}$, for $k = 1, \ldots, r$, are related to the matrix filters $\boldsymbol{H}$ and $\boldsymbol{G}$ by

$$\begin{aligned} h_k(\ell r+j-1) &= h_{k,j}(\ell), \\ g_k(\ell r+j-1) &= g_{k,j}(\ell), \qquad (11) \end{aligned}$$

where $\ell \in \mathbb{Z}, j = 1, 2, ..., r$, and the multiplexer operation, **MUX**, is given by

$$v(rn+j-1) = x_j(n), \qquad (12)$$

for $n \in \mathbb{Z}$, and $j = 1, 2, ..., r$. Denote $\widehat{\boldsymbol{h}}_k(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k(n)e^{-j\omega n}$ and $\widehat{\boldsymbol{g}}_k(\omega) := \frac{1}{2}\sum_{k\in\mathbb{Z}} g_k(n)e^{-j\omega n}$, $k = 1, 2, \ldots, r$. We can further show that

$$\begin{aligned} \widehat{\boldsymbol{h}}_s(\omega) &= \sum_{k\in\mathbb{Z}}\left(\sum_{t=1}^{r} h_{s,t}(k)e^{-j(t-1)\omega}\right)e^{-jrk\omega} \\ &= \sum_{t=1}^{r}\left(\sum_{k\in\mathbb{Z}} h_{s,t}(k)e^{-jrk\omega}\right)e^{-j(t-1)\omega}. \end{aligned}$$

Hence, it is shown that any multifilter with a multiplicity r can be sufficiently represented by a set of r equivalent scalar filters, each consisting of r polyphases, where the t^{th} polyphase of the s^{th} equivalent scalar filter is given by $\sum_{k\in\mathbb{Z}} h_{s,t}(k)e^{-jrk\omega}$, for all $s, t = 1, 2, \ldots, r$. A similar relationship also holds for $\widehat{\boldsymbol{g}}_k(\omega)$. In summary, the equivalent system of scalar filters guarantees an identical MIMO relationship with that of the multifilters.

The above relationship provides us with a new framework to design a multifilter system by imposing desirable filter properties on the set of equivalent scalar filters. This motivates the idea of GMPs. For simplicity of exposition, but without loss of generality, we consider only multiwavelets with multiplicity $r = 2$. A given multiwavelet system with multiplicity 2 is said to possess a GMP order (d_1, d_2, e_1) if the equivalent lowpass and highpass scalar filters, $\boldsymbol{h}_k$ and $\boldsymbol{g}_k$, for $k = 1, 2$, satisfy the following GMP conditions:

(i) $\widehat{\boldsymbol{h}}_k^{(\nu)}(0) = \delta_{\nu,0}, \quad \nu = 0, 1, \ldots, d_1 - 1,$

(ii) $\widehat{\boldsymbol{h}}_k^{(\nu)}(\pi) = 0, \quad \nu = 0, 1, \ldots, d_2 - 1,$

(iii) $\widehat{\boldsymbol{g}}_k^{(\nu)}(0) = 0, \quad \nu = 0, 1, \ldots, e_1 - 1,$

where the superscript $^{(\nu)}$ denotes the ν^{th}-order derivative, and d_1, d_2, e_1 are positive integers. These conditions ensure that $\boldsymbol{h}_k$ and $\boldsymbol{g}_k$ behave as lowpass and highpass filters, respectively.

Now it will be useful to understand the idea of GMPs *directly* in terms of some properties of the matrix filters. Specifically, we will investigate the eigenvector characteristics of matrix filters that correspond to multiwavelets possessing GMPs. Since $\widehat{\boldsymbol{H}}(0)$ satisfies Condition E and has a vanishing moment of at least order one, there exists a vector $\boldsymbol{v}$ such that

$$\boldsymbol{v}^T \widehat{\boldsymbol{H}}(\pi\nu) = \delta_{\nu,0} \boldsymbol{v}^T, \quad \nu \in \mathbb{Z}/2\mathbb{Z}, \qquad (13)$$

where $\delta_{m,n} = 1$ only if $m = n$; otherwise, it is equal to 0. By setting $\omega = 0$ in the CQF relation (5) and multiplying it with $\boldsymbol{v}^T$ from the left sides, we have

$$\boldsymbol{v}^T \widehat{\boldsymbol{H}}(0) \widehat{\boldsymbol{H}}(0)^* + \boldsymbol{v}^T \widehat{\boldsymbol{H}}(\pi) \widehat{\boldsymbol{H}}(\pi)^* = \boldsymbol{v}^T. \quad (14)$$

Clearly, by applying Eq. (13) into (14), $\boldsymbol{v}$ is a right eigenvector of $\widehat{\boldsymbol{H}}(0)$ corresponding to an eigenvalue $\lambda = 1$. Since $\lambda = 1$ is a simple eigenvalue of $\widehat{\boldsymbol{H}}(0)$ and the multiwavelet system possesses GMP condition (i), we have $\boldsymbol{v} = (1, 1)^T$. Interestingly, from Eq. (3), we have, up to a constant multiple,

$$\widehat{\boldsymbol{\Phi}}(0) = \boldsymbol{v} = (1, 1)^T. \qquad (15)$$

Furthermore, since the system also possesses GMP condition (ii), it implies that $\widehat{\boldsymbol{H}}(0)$ has an eigenvector $(1, -1)^T$ corresponding to an eigenvalue $\lambda = 0$, which subsequently implies that the matrix $\widehat{\boldsymbol{H}}(0)$ is singular. From Eq. (6) and the GMP condition (iii), it is clear that $\boldsymbol{v} = (1, 1)^T$ is also an eigenvector of $\widehat{\boldsymbol{G}}(0)$ corresponding to an eigenvalue $\lambda = 0$. Therefore by analyzing the eigenvector properties of the matrix refinement masks, we can gain an insight into the GMPs of a multiwavelet system. A step-by-step procedure for determining the GMP order of a multifilter system is given in [7].

3 Multiwavelet Filter Construction and Initialization

In this section we show that the proposed idea of GMPs does not only provide a useful tool to analyze a multifilter system, but it can also result in the construction of new classes of symmetric-antisymmetric orthogonal multiwavelets (SAOMWs) and symmetric-antisymmetric biorthogonal multiwavelets (SABMWs) that are suitable for image compression. Next, we will address the problem of multiwavelet initialization or prefiltering, which is indirectly also motivated by the idea of GMPs.

3.1 Construction of Multiwavelets with GMPs

It is noted that Eq. (15) imposes a rather restrictive condition on the design of multiwavelets. However, it is well-known that we can perform a change of basis by applying a similarity transformation to the matrix filters. Let $\boldsymbol{\Phi}^\sharp = \boldsymbol{U}\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}^\sharp = \boldsymbol{U}\boldsymbol{\Psi}$ be the *new*[1] multiscaling and multiwavelet function vectors with the corresponding new matrix filters

$$\boldsymbol{H}^\sharp := \boldsymbol{U}\boldsymbol{H}\boldsymbol{U}^T, \qquad \boldsymbol{G}^\sharp := \boldsymbol{U}\boldsymbol{G}\boldsymbol{U}^T.$$

Because of the GMP requirement in Eq. (15), we choose an orthogonal matrix $\boldsymbol{U}$ such that $\widehat{\boldsymbol{\Phi}}^\sharp(0)$ is parallel to vector $(1, 1)^T$. This leads us to the following relation:

$$\widehat{\boldsymbol{H}}(0)\boldsymbol{U}^{-1}(1, 1)^T = \boldsymbol{U}^{-1}(1, 1)^T. \qquad (16)$$

Note that such a similarity transformation still guarantees that $\{\boldsymbol{H}^\sharp, \boldsymbol{G}^\sharp\}$ also satisfies the PR criteria (5)–(7). Therefore we can say that

[1] For the rest of the paper, the superscript $^\sharp$ will denote the new (similarly transformed) version of a given multiwavelet filter bank system.

$\{\boldsymbol{H}, \boldsymbol{G}\}$ possesses a GMP order (d_1, d_2, e_1) if $\{\boldsymbol{H}^\sharp, \boldsymbol{G}^\sharp\}$ possesses a GMP order (d_1, d_2, e_1).

The example below shows a class of length-4 multiwavelets possessing GMPs that are used in our image compression simulations later. The matrix lowpass filters, when parameterized by α, are given as follows:

$$\boldsymbol{H}_0 = \begin{pmatrix} \frac{1}{\alpha^2+1} & \frac{\alpha}{\alpha^2+1} \\ \frac{1}{\alpha^2+1} & \frac{-\alpha}{\alpha^2+1} \end{pmatrix},$$

$$\boldsymbol{H}_1 = \begin{pmatrix} \frac{\alpha^2}{\alpha^2+1} & \frac{\alpha}{\alpha^2+1} \\ \frac{-\alpha^2}{\alpha^2+1} & \frac{\alpha}{\alpha^2+1} \end{pmatrix}, \qquad (17)$$

$\boldsymbol{H}_2 = \boldsymbol{S}\boldsymbol{H}_1\boldsymbol{S}$, and $\boldsymbol{H}_3 = \boldsymbol{S}\boldsymbol{H}_0\boldsymbol{S}$, where $\boldsymbol{S} = diag(1,1)$. The corresponding matrix highpass filters are given by $\boldsymbol{G}_k = (-1)^{k-1}\boldsymbol{H}_{3-k}\boldsymbol{J}$, for $k = 0, 1, 2, 3$, where $\boldsymbol{J}$ is a reversal matrix, i.e., $\boldsymbol{J} = antidiag(1,1)$. By varying the parameter α, we can obtain multiwavelets with different properties. For example, when $\alpha = 4 + \sqrt{19}$, the multiwavelet SA4(1) has an approximation order of two and a GMP order (1,1,1); when $\alpha = 4 + \sqrt{15}$, the multiwavelet SA4(2) has an approximation order of one and a GMP order (1,2,1); and when $\alpha = 6.981578516$, the multiwavelet SA4(3) has an approximation order of one, a GMP order (1,1,1), and the sharpest cuttoff frequency.

The idea of GMP has also been extended to constructing new biorthogonal multiwavelets (SABMWs) [5]. As examples to demonstrate their image compression performance, three members from this family will be analyzed in Sec. 4. They are the SA(4/4), SA(5/5), and SA(9/7), in which the numbers in brackets represent the supports of the primal and dual matrix lowpass filters, respectively.

3.2 Multiwavelet Initialization

Another topic that is of critical importance for successful application of multiwavelets is the problem of multiwavelet initialization or pre-filtering. Several interesting proposals have been reported to address this problem (e.g. [10]), but they, however, do have several drawbacks such as over-sampling of the input data. In this subsection, we will present an efficient and general solution for multiwavelet pre-filtering that is orthogonal and it provides a non-redundant[2] representation of the input signal. Motivated by the proposed concepts of GMPs and equivalent scalar filters, we have designed the pre-filtering process in such a way so that it works in tandem with multifilters possessing GMPs.

Recall from Fig. 1 (b) that the multiplexed stream $\boldsymbol{v}$ is fed independently into each of the equivalent scalar filters. It is conceptually identical to consider $\boldsymbol{v}$ as the given input signal that needs to be *demultiplexed* into multiple input streams, $\boldsymbol{x}$, before they are decomposed further by the designed multifilters. This process of generating the multiple input streams from a single stream is called *multiwavelet initialization* or *pre-filtering*. The relationship that defines the demultiplexer operation is given in Eq. (12) as a dual of the multiplexer operation. For the case of multiplicity $r = 2$, we essentially pair up the given signal $\boldsymbol{v}$ into input vector streams, which can be assumed to be locally constant. From Eq. (16), it becomes clear that the appropriate pre-filter is then given by $\boldsymbol{U}^{-1}$, where

$$\boldsymbol{U} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix},$$

with $\theta = -\frac{\pi}{4}$ when $\widehat{\phi}_1(0) = 0$, or otherwise $\theta = \frac{\pi}{4} - \tan^{-1}\left(\frac{\widehat{\phi}_2(0)}{\widehat{\phi}_1(0)}\right)$ for $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4}]$. A procedure for choosing the better pre-filter $\boldsymbol{U}^{-1}$ is provided in [7]. For all the examples of multiwavelets used in the simulations, we employed the first pre-filter of Eq. (18).

In summary, the following few points about the proposed pre-filtering technique are worth highlighting. First, the pre-filter is very simple and hence *efficient* in terms of low computational complexity. Specifically, for our classes of symmetric-antisymmetric orthogonal and biorthogonal multiwavelets, the pre-filter can be expressed as

$$\boldsymbol{U}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad (18)$$

which involves practically no arithmetic computations at all since the normalization factor

[2]Such a compact representation is critical to applications such as image compression.

$1/\sqrt{2}$ can be absorbed into the first level of multiwavelet decomposition. Second, although the proposed multiwavelet initialization framework is motivated by the idea of GMPs, it is very *robust* in the sense that it is applicable to any multifilter system, regardless of whether it possesses GMPs or not. Third, it is obvious that $\boldsymbol{U}^{-1}$ is *orthogonal*; this ensures that the orthogonality and approximation order of the designed multifilters are well preserved after pre-filtering. Finally, it is noted that, after pre-filtering, we actually apply the original multifilters that are symmetric/antisymmetric, instead of the new multifilters which may have lost their linear phase property after similarity transformation. This allows us to employ symmetric extension techniques [9] (which give better results than using periodic extension) at the image boundaries because both SAOMWs and SABMWs are linear phase multifilters. A detailed explanation, together with illustrations, on how to integrate the proposed pre-filtering technique with multiresolution image decomposition/reconstruction is given in [7].

4 Performance Analysis

In this section, we will investigate the performance of both SAOMWs and SABMWs, and the proposed pre-filtering technique in image compression application. Five multifilters, namely, SA4(1), SA4(3), SA(4/4), SA(5/5), and SA(9/7) were used in our simulations. We also included the results using the following scalar wavelets: Daubechies' orthogonal maxflat 8-tap filter (D8), Daubechies' biorthogonal 9/7 filter (D(9/7)) [1], and Villasenor's biorthogonal 18/10 filter (V(18/10)) [8]. For a fair comparison of the contributions of different filters, we have employed the *same* still image codec [4] in all the simulations (other codecs such as [6] also gave similar relative performances).

Table 1 shows the peak signal-to-noise ratios (PSNRs) of the reconstructed images. Four standard monochrome images, namely, Lena, Barbara, Boat, and Goldhill have been tested over five different compression ratios. The bold value in each row of the table indicates the best PSNR value for a particular CR-image pair. It is evident that SAOMWs have consistently outperformed the scalar wavelet D8 by a significant margin of up to 0.88 dB. Our SABMWs have also performed better than D(9/7), one of the most widely used scalar wavelets for image compression, and V(18/10) by up to 0.75 dB and 0.6 dB, respectively. Subjective performance comparisons are illustrated in Fig. 2. It is clear that the designed multifilters possessing GMPs can better preserve the texture of the table cloth and trousers in the Barbara image.

In addition to improved image compression performance, we also achieved lower computational complexity [5]. For example, the application of conventional octave-scale decomposition using scalar wavelets D8, D(9/7), and V(18/10) will demand a higher computational cost by a factor of 2.67, 1.5, and 2.33, respectively, when compared with using the combination of the multiwavelet SA(4/4) and the proposed efficient pre-filtering framework.

5 Conclusions

We introduced a new tool called *"good multifilter properties"* (GMPs) that can provide us with better insights into the analysis, construction, and application of multiwavelet filters. For analysis, we established the concept of an *equivalent system of scalar filters*, which provides a sufficient and equivalent representation of the multiple-input multiple-output relationship of a given multifilter system. Two classes of symmetric-antisymmetric orthogonal and biorthogonal multiwavelets possessing GMPs were also constructed. For efficient application of multiwavelets, we proposed an orthogonal pre-filtering framework that can maintain the compact representation of a given input signal. Finally, extensive image compression simulations confirmed that our proposals can provide significant improvements in both objective and subjective image compression performance, and lower total computational requirement, when compared with us-

Image	CR	D8	D(9/7)	V(18/10)	SA4(1)	SA4(3)	SA(4/4)	SA(5/5)	SA(9/7)
Lena	8:1	40.57	41.01	41.08	40.85	40.87	41.01	41.08	**41.18**
	16:1	37.14	37.83	37.93	37.54	37.59	37.84	37.96	**38.04**
	32:1	33.88	34.74	34.83	34.42	34.47	34.77	34.88	**34.98**
	64:1	30.84	31.75	31.86	31.46	31.50	31.83	31.85	**31.96**
	128:1	28.18	29.04	29.08	28.75	28.78	29.05	29.05	**29.09**
Barbara	8:1	36.73	37.45	38.02	37.25	37.43	37.71	37.82	**38.09**
	16:1	31.53	32.10	32.50	32.13	32.27	32.46	32.56	**32.85**
	32:1	27.70	28.13	28.32	28.30	28.39	28.55	28.53	**28.72**
	64:1	25.02	25.38	25.30	25.56	25.59	25.56	25.68	**25.90**
	128:1	23.53	23.77	23.78	23.96	23.98	**24.05**	23.97	24.02
Boat	8:1	38.48	39.11	39.23	39.12	39.20	39.43	39.42	**39.44**
	16:1	33.84	34.45	34.71	34.49	34.55	34.86	34.83	**34.94**
	32:1	30.28	30.97	31.05	30.87	30.93	31.15	31.15	**31.25**
	64:1	27.60	28.16	28.18	28.10	28.12	28.31	28.30	**28.33**
	128:1	25.48	25.90	25.95	25.83	25.85	25.99	25.99	**26.01**
Goldhill	8:1	36.10	36.55	36.67	36.57	36.61	36.73	36.69	**36.83**
	16:1	32.61	33.13	33.19	33.14	33.18	33.29	33.26	**33.42**
	32:1	29.97	30.56	30.63	30.60	30.62	**30.77**	30.73	30.75
	64:1	27.88	28.48	28.56	28.52	28.55	**28.68**	28.65	28.66
	128:1	25.91	26.73	26.81	26.76	26.79	26.88	26.86	**26.96**

Table 1: Image compression performance comparisons of (bi)orthogonal scalar filters D8, D(9/7) and V(18/10), and (bi)orthogonal multifilters SA4(1), SA4(3), SA(4/4), SA(5/5), and SA(9/7) for four 512×512 monochrome images and at five different compression ratios (CR).

ing some popular scalar wavelets. Specifically, some members of our designed multiwavelets possessing GMPs can outperform the scalar wavelets Daubechies' biorthogonal 9/7 and Villasenor's biorthogonal 18/10 by up to 0.75 dB and 0.6 dB, respectively.

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Figure 2: Barbara image: (a) original; (c) using D(9/7) at 32:1; (e) using SA(4/4) at 32:1; (b),(d),(f) zoom-in versions of (a), (c) and (e), respectively.