Symmetric-Antisymmetric Orthonormal Multiwavelets and Related Scalar Wavelets ${ }^{1}$

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For compactly supported symmetric-antisymmetric orthonormal multiwavelet systems with multiplicity 2 , we first show that any length $-2 N$ multiwavelet system can be constructed from a length- $(2 N+1)$ multiwavelet system and vice versa. Then we present two explicit formulations for the construction of multiwavelet functions directly from their associated multiscaling functions. This is followed by the relationship between these multiscaling functions and the scaling functions of related orthonormal scalar wavelets. Finally, we present two methods for constructing families of symmetric-antisymmetric orthonormal multiwavelet systems via the construction of the related scalar wavelets. © 2000 Academic Press

## 1. INTRODUCTION

The study of multiwavelets was first initiated by Goodman et al. [8] in 1993, and since then multiwavelets have received considerable attention from the wavelets research communities both in theory $[1,2,6-11,15-18]$ and in applications such as signal compression and denoising $[19,21,24]$. The main motivation for multiwavelets is that they can simultaneously possess desirable properties such as symmetry, orthogonality, and shorter support for a given approximation order, which are not possible in any realvalued scalar wavelet [4]. One of the earliest and most popularly used multiwavelets with multiplicity 2 is the GHM multiwavelet which was constructed by Geronimo et al. $[6,7]$ using fractal interpolation. The multiscaling functions of the GHM multiwavelet are both symmetric and orthonormal. Later, by imposing Hermite interpolating conditions, Chui and Lian [1] constructed symmetry-antisymmetric orthonormal multiwavelets with particular emphasis on the maximum order of polynomial reproduction and gave examples for length-3 and length-4 multiwavelets. In our preceding paper [21], we introduced another class of symmetric-antisymmetric orthonormal multiwavelets which possess a new

[^0]property called the good multifilter properties (GMP) and demonstrated that they can be useful for image compression. In this paper, we will further the study of this class of multiwavelets and expound its relationship with related orthonormal scalar wavelets.

We begin with some basic theory and notations to be used throughout this paper. For a multiwavelet system with multiplicity $r$, the vector $\phi(x)=\left(\phi_{1}(x), \ldots, \phi_{r}(x)\right)^{T}$ is a compactly supported orthonormal scaling vector generating a multiresolution analysis (MRA) $\left\{V_{j}\right\}_{j \in \mathbb{Z}}$ of $L^{2}(\mathbb{R})$ with

$$
\cdots \subset V_{-1} \subset V_{0} \subset V_{1} \subset \cdots \subset L^{2}(\mathbb{R}),
$$

where $V_{j}:=\left\{f: f\left(2^{-j}.\right) \in V_{0}\right\}, j \in \mathbb{Z}$. The vector $\phi$ has the following properties:

- $\phi$ satisfies a refinement equation

$$
\begin{equation*}
\phi(x)=\sum_{k \in \mathbb{Z}} \boldsymbol{P}_{k} \phi(2 x-k), \tag{1.1}
\end{equation*}
$$

for some finite sequence $\left\{\boldsymbol{P}_{k}\right\}$ of $r \times r$ matrices.

- The integer shifts $\left\{\phi_{i}(\cdot-k): k \in \mathbb{Z}, i=1, \ldots, r\right\}$ constitute an orthonormal basis of $V_{0}$.

Associated with $\phi$ is an orthonormal multiwavelet vector $\psi(x)=\left(\psi_{1}(x), \ldots, \psi_{r}(x)\right)^{T}$ with the following properties:

- There exists a finite sequence $\left\{\boldsymbol{Q}_{k}\right\}$ of $r \times r$ matrices such that

$$
\begin{equation*}
\psi(x)=\sum_{k \in \mathbb{Z}} \boldsymbol{Q}_{k} \phi(2 x-k) \tag{1.2}
\end{equation*}
$$

- The integer shifts $\left\{\psi_{i}(\cdot-k): k \in \mathbb{Z}, i=1, \ldots, r\right\}$ constitute an orthonormal basis of $W_{0}$, where $W_{0}$ is the orthogonal complement of $V_{0}$ in $V_{1}$.

We will refer to $\phi_{i}$ 's and $\psi_{i}$ 's as the multiscaling and multiwavelet functions, respectively, and the matrix sequences $\left\{\boldsymbol{P}_{k}\right\}$ and $\left\{\boldsymbol{Q}_{k}\right\}$ as the lowpass and highpass sequences, respectively. We also say that the pair $\left\{\boldsymbol{P}_{k}, \boldsymbol{Q}_{k}\right\}$ (or $\{\phi, \psi\}$ ) generates an orthonormal multiwavelet system.

The Fourier transforms of sequences $\left\{\boldsymbol{P}_{k}\right\}$ and $\left\{\boldsymbol{Q}_{k}\right\}$, i.e., $\boldsymbol{P}(\omega):=\frac{1}{2} \sum_{k \in \mathbb{Z}} \boldsymbol{P}_{k} e^{-j k \omega}$ and $\boldsymbol{Q}(\omega):=\frac{1}{2} \sum_{k \in \mathbb{Z}} \boldsymbol{Q}_{k} e^{-j k \omega}, j=\sqrt{-1}$, will be referred to as the refinement mask and the wavelet mask, respectively. The orthonormality of $\phi$ and $\psi$ implies the following perfect reconstruction (PR) conditions,

$$
\begin{align*}
\boldsymbol{P}(\omega) \boldsymbol{P}^{*}(\omega)+\boldsymbol{P}(\omega+\pi) \boldsymbol{P}^{*}(\omega+\pi) & =\boldsymbol{I}_{r \times r}  \tag{1.3}\\
\boldsymbol{P}(\omega) \boldsymbol{Q}^{*}(\omega)+\boldsymbol{P}(\omega+\pi) \boldsymbol{Q}^{*}(\omega+\pi) & =\mathbf{0}_{r \times r}  \tag{1.4}\\
\boldsymbol{Q}(\omega) \boldsymbol{Q}^{*}(\omega)+\boldsymbol{Q}(\omega+\pi) \boldsymbol{Q}^{*}(\omega+\pi) & =\boldsymbol{I}_{r \times r} \tag{1.5}
\end{align*}
$$

where the superscript * denotes the conjugate transpose. In addition, the above three equations are equivalent to the following equations,

$$
\begin{equation*}
\sum_{k \in \mathbb{Z}} \boldsymbol{P}_{k} \boldsymbol{P}_{k+2 i}^{T}=2 \delta_{i, 0} \boldsymbol{I}_{r \times r} \tag{1.6}
\end{equation*}
$$

$$
\begin{align*}
& \sum_{k \in \mathbb{Z}} \boldsymbol{P}_{k} \boldsymbol{Q}_{k+2 i}^{T}=\mathbf{0}_{r \times r}  \tag{1.7}\\
& \sum_{k \in \mathbb{Z}} \boldsymbol{Q}_{k} \boldsymbol{Q}_{k+2 i}^{T}=2 \delta_{i, 0} \boldsymbol{I}_{r \times r} \tag{1.8}
\end{align*}
$$

for $i \in \mathbb{Z}$, where

$$
\delta_{i, j}= \begin{cases}1, & i=j \\ 0, & \text { otherwise } .\end{cases}
$$

Specifically, the sequence $\left\{\boldsymbol{P}_{k}\right\}$ which satisfies (1.3) or (1.6) is called a conjugate quadrature filter (CQF). In order to distinguish it from the usual CQF in the scalar setting, we will refer to $\left\{\boldsymbol{P}_{k}\right\}$ as a matrix CQF throughout this paper.

The transition operator for $\boldsymbol{P}(\omega)$ is defined as

$$
\begin{equation*}
\boldsymbol{T}_{\boldsymbol{P}} \boldsymbol{H}(\omega):=\boldsymbol{P}\left(\frac{\omega}{2}\right) \boldsymbol{H}\left(\frac{\omega}{2}\right) \boldsymbol{P}^{*}\left(\frac{\omega}{2}\right)+\boldsymbol{P}\left(\frac{\omega}{2}+\pi\right) \boldsymbol{H}\left(\frac{\omega}{2}+\pi\right) \boldsymbol{P}^{*}\left(\frac{\omega}{2}+\pi\right) . \tag{1.9}
\end{equation*}
$$

This operator is useful for characterizing the orthonormality of $\phi$. The refinement function vector $\phi$ is orthonormal if and only if $\left\{\boldsymbol{P}_{k}\right\}$ is a matrix CQF and its transition operator $\boldsymbol{T}_{\boldsymbol{P}}$ satisfies Condition E (see [17]). We say that a square matrix $\boldsymbol{M}$ (or a linear operator) satisfies Condition E if its spectral radius $\rho(\boldsymbol{M}) \leq 1$ with 1 being the only eigenvalue of $\boldsymbol{M}$ on the unit circle and it is simple.

In this paper, we will focus on a class of symmetric-antisymmetric orthonormal multiwavelet systems with multiplicity $r=2$, whose members have finite and real-valued lowpass sequences $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{L}$ satisfying the following:

$$
\begin{align*}
& \boldsymbol{P}_{0} \text { and } \boldsymbol{P}_{L} \text { are nonzero matrices }  \tag{1.10}\\
& \boldsymbol{P}_{k}=\boldsymbol{S} \boldsymbol{P}_{L-k} \boldsymbol{S}, \quad k=0,1, \ldots, L, \quad \text { where } \boldsymbol{S}=\operatorname{diag}(1,-1)  \tag{1.11}\\
& \boldsymbol{P}(0)=\left[\begin{array}{ll}
1 & 0 \\
0 & \lambda
\end{array}\right], \quad|\lambda|<1 . \tag{1.12}
\end{align*}
$$

Collectively, we refer to the above conditions as Condition SA for easy referencing.
The second condition (1.11) implies that the corresponding multiscaling functions form a symmetric-antisymmetric pair as shown in the following [1]:

$$
\begin{align*}
\boldsymbol{P}_{k} & =\boldsymbol{S} \boldsymbol{P}_{L-k} \boldsymbol{S}, \quad k=0,1, \ldots, L \\
& \Longleftrightarrow \quad \boldsymbol{P}(\omega)=\boldsymbol{S P}(-\omega) \boldsymbol{S} e^{-j L \omega} \\
& \Longleftrightarrow \quad \phi_{i}(x)=(-1)^{i-1} \phi_{i}(L-x), \quad i=1,2 \tag{1.13}
\end{align*}
$$

The orthonormality of $\phi$ also implies that $\widehat{\phi_{1}}(0)=1$ and $\widehat{\phi_{2}}(0)=0$.
The third condition (1.12) is a necessary condition [3,13] for any lowpass sequence satisfying (1.10) and (1.11) (or the corresponding multiscaling function vector) to generate a MRA.

Note that for $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{L}$, the transition operator $\boldsymbol{T}_{\boldsymbol{P}}$ is a linear operator on $\mathbb{H}_{L}$, where $\mathbb{H}_{L}$ is the space of all $r \times r$ matrices whose entries are trigonometric polynomials such that their Fourier coefficients are supported in $[-L, L]$.

We shall adopt the following notations and assumptions throughout for ease of exposition. First we define the following orthogonal matrices:

$$
\boldsymbol{U}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right], \quad \boldsymbol{S}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right], \quad \boldsymbol{A}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] .
$$

There are frequent occasions where we apply similarity transformations to given matrices using the transition matrix $\boldsymbol{U}$ and then the reverse transformations. We shall use the superscript ${ }^{\sharp}$ for denoting the matrix obtained by applying the similarity transformation with $\boldsymbol{U}$ as the transition matrix. The same superscript will be used to represent the elements of the resultant matrix as well as the multiscaling functions and the multiwavelet functions. For example, we have

$$
\begin{equation*}
\boldsymbol{P}_{k}^{\sharp}=\boldsymbol{U} \boldsymbol{P}_{k} \boldsymbol{U}^{-1}, \tag{1.14}
\end{equation*}
$$

where $\boldsymbol{P}_{k}=\left(p_{i j}(k)\right)_{i, j=1}^{2}$ and $\boldsymbol{P}_{k}^{\sharp}=\left(p_{i j}^{\sharp}(k)\right)_{i, j=1}^{2}$.
The rest of the paper is organised as follows. In Section 2, we establish several results on matrix CQFs, in particular, the relationship between even- and odd-length matrix CQFs satisfying Condition SA. In Section 3, two explicit formulations which can be used to derive the highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}$ directly from the corresponding lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}$ are given. In Section 4, the relationship between orthonormal scalar wavelets and a class of symmetric-antisymmetric orthonormal multiwavelet systems is first established. We then provide a procedure for constructing families of multiwavelet systems from related scalar wavelets with examples given for length- 4 and length- 6 multiwavelet systems.

## 2. SOME RESULTS ON MATRIX CQFs

The purpose of this section is to present several results on matrix CQFs satisfying Condition SA. In particular we give an intrinsic characterization of the relationship between even- and odd-length matrix CQFs. We will prove that length- $2 N$ and length$(2 N+1)$ matrix CQFs satisfying Condition SA can be obtained from one another. To this end, we find it convenient to change the matrix CQF and its refinement mask using the orthogonal matrix $\boldsymbol{U}$. First of all, as $\boldsymbol{U}$ is orthogonal, it is clear [1] that if $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{L}$ is matrix CQF, then $\left\{\boldsymbol{P}_{k}^{\sharp}\right\}_{k=0}^{L}$ is also a matrix CQF.

Noting that $\boldsymbol{A}=\boldsymbol{U} \boldsymbol{S} \boldsymbol{U}^{-1}$, the following lemma on the matrix sequences $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{L}$ and $\left\{\boldsymbol{U}_{k}^{\sharp}\right\}_{k=0}^{L}$ and their corresponding masks can be easily established.

Lemma 1. The following four statements are equivalent:
(i) $\boldsymbol{P}_{k}=\boldsymbol{S} \boldsymbol{P}_{L-k} \boldsymbol{S}, k=0,1, \ldots, L$.
(ii) $\boldsymbol{P}_{k}^{\sharp}=\boldsymbol{A} \boldsymbol{P}_{L-k}^{\sharp} \boldsymbol{A}, k=0,1, \ldots, L$.
(iii) $\boldsymbol{P}(\omega)=\boldsymbol{S} \boldsymbol{P}(-\omega) \boldsymbol{S} e^{-j L \omega}$.
(iv) $\boldsymbol{P}^{\sharp}(\omega)=\boldsymbol{A} \boldsymbol{P}^{\sharp}(-\omega) \boldsymbol{A} e^{-j L \omega}$.

From a given matrix CQF, $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{L}$, satisfying Condition SA, one can generate other matrix CQFs satisfying Condition SA. The following lemma gives three such possible ways.

Lemma 2. Let $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{L}$ be a matrix CQF satisfying Condition SA. Then each of the following matrix sequences,
(i) $\left\{\boldsymbol{S} \boldsymbol{P}_{k}\right\}_{k=0}^{L}$,
(ii) $\left\{\boldsymbol{P}_{k} \boldsymbol{S}\right\}_{k=0}^{L}$,
(iii) $\left\{\boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{S}\right\}_{k=0}^{L}$,
also forms a matrix CQF satisfying Condition SA.
Proof. Here we show the proof for case (i). The proof for the other two cases is similar. Let $\boldsymbol{R}_{k}:=\boldsymbol{S} \boldsymbol{P}_{k}, k=0,1, \ldots, L$. Then

$$
\sum_{k=0}^{L-2 i} \boldsymbol{R}_{k}\left(\boldsymbol{R}_{k+2 i}\right)^{T}=\boldsymbol{S}\left(\sum_{k=0}^{L-2 i} \boldsymbol{P}_{k} \boldsymbol{P}_{k+2 i}^{T}\right) \boldsymbol{S}=2 \delta_{i, 0} \boldsymbol{I}_{2 \times 2}, \quad i \in \mathbb{Z}
$$

which means that $\left\{\boldsymbol{R}_{k}\right\}_{k=0}^{L}$ forms a matrix CQF.
Next, the sequence $\left\{\boldsymbol{R}_{k}\right\}_{k=0}^{L}$ clearly satisfies condition (1.10). For condition (1.11), we have

$$
\boldsymbol{R}_{k}=\boldsymbol{S} \boldsymbol{P}_{k}=\boldsymbol{S}\left(\boldsymbol{S} \boldsymbol{P}_{L-k} \boldsymbol{S}\right)=\boldsymbol{S} \boldsymbol{R}_{L-k} \boldsymbol{S}, \quad k=0,1, \ldots, L
$$

Finally, $\boldsymbol{R}(\omega)=\boldsymbol{S P}(\omega)$ and

$$
\boldsymbol{R}(0)=\boldsymbol{S} \boldsymbol{P}(0)=\left[\begin{array}{cc}
1 & 0 \\
0 & -\lambda
\end{array}\right]
$$

where $|\lambda|<1$. Thus, Condition (1.12) is satisfied. The matrix $\operatorname{CQF}\left\{\boldsymbol{R}_{k}\right\}_{k=0}^{L}$ satisfies Condition SA as a result.

Case (iii) in the above lemma is actually obtained by applying a similarity transformation of each $\boldsymbol{P}_{k}$ with $\boldsymbol{S}$ as the transition matrix. This corresponds to reversing the order of matrix coefficients in the matrix CQF.

Next we will establish the relationship between the even-length and odd-length matrix CQFs satisfying Condition SA. Before we proceed further, consider the following lemma.

Lemma 3. Let $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N}$ be an odd-length lowpass sequence satisfying Condition SA. Then one of the following statements holds,

$$
\text { (i) } \boldsymbol{P}_{0}^{\sharp}=\left[\begin{array}{ll}
0 & \alpha_{1} \\
0 & \alpha_{2}
\end{array}\right], \quad \text { or } \quad \text { (ii) } \quad \boldsymbol{P}_{0}^{\sharp}=\left[\begin{array}{ll}
\alpha_{2} & 0 \\
\alpha_{1} & 0
\end{array}\right] \text {, }
$$

where $\alpha_{1}=p_{11}(0)-p_{21}(0)$ and $\alpha_{2}=p_{11}(0)+p_{21}(0)$.
Proof. The CQF condition (1.6) and the Condition SA imply $\boldsymbol{P}_{0} \boldsymbol{P}_{2 N}^{T}=\mathbf{0}_{2 \times 2}$ and $\boldsymbol{P}_{2 N}=\boldsymbol{S} \boldsymbol{P}_{0} \boldsymbol{S}$, respectively. Consequently, we have

$$
p_{11}^{2}(0)=p_{12}^{2}(0), \quad p_{21}^{2}(0)=p_{22}^{2}(0), \quad \text { and } \quad p_{11}(0) p_{21}(0)-p_{12}(0) p_{22}(0)=0
$$

Clearly, there are two possible cases:

- $p_{12}(0)=p_{11}(0), p_{22}(0)=p_{21}(0)$. This implies $\boldsymbol{P}_{0}^{\sharp}$ has the form in (i).
- $p_{12}(0)=-p_{11}(0), p_{22}(0)=-p_{21}(0)$. This implies $\boldsymbol{P}_{0}^{\sharp}$ has the form in (ii).

Note that if we precede the similarity transformation of $\boldsymbol{P}_{0}$ using the matrix $\boldsymbol{U}$ with another similarity transformation using transition matrix $\boldsymbol{S}$, i.e., $\boldsymbol{P}_{0}^{\sharp}=\boldsymbol{U} \boldsymbol{S} \boldsymbol{P}_{0} \boldsymbol{S} \boldsymbol{U}^{-1}$, then applying the result for the second case actually gives rise to a $\boldsymbol{P}_{0}^{\sharp}$ of the form in Lemma 3(i). Hence, we will say that an orthonormal multiscaling function vector $\phi$ is unique if its lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}$ is unique up to the similarity transformation of $\boldsymbol{P}_{k}$ 's with one of the following three orthogonal matrices $\boldsymbol{S},-\boldsymbol{S}$, or $-\boldsymbol{I}_{2 \times 2}$. In what follows, for odd-length orthonormal multiwavelet systems satisfying Condition SA, we always assume that $\boldsymbol{P}_{0}^{\sharp}$ has the form in Lemma 3(i).

Let $\boldsymbol{E}_{a, b}$ denote the matrix $\boldsymbol{e}_{a} \boldsymbol{e}_{b}^{T}, a, b \in[1,2]$, and $\boldsymbol{e}_{k}$ denote the $k$ th unit column 2 -vector.

THEOREM 1. Let $\left\{\boldsymbol{P}_{e, k}\right\}_{k=0}^{2 N-1}$ be an even-length matrix CQF which satisfies Condition SA. Construct the matrix sequence $\left\{\boldsymbol{P}_{o, k}^{\sharp}\right\}_{k=0}^{2 N}$ from $\left\{\boldsymbol{P}_{e, k}^{\sharp}\right\}_{k=0}^{2 N-1}$ using the following rules:

$$
\boldsymbol{P}_{o, k}^{\sharp}= \begin{cases}\boldsymbol{P}_{e, 0}^{\sharp} \boldsymbol{E}_{1,2}, & k=0  \tag{2.1}\\ \boldsymbol{P}_{e, k-1}^{\sharp} \boldsymbol{E}_{2,1}+\boldsymbol{P}_{e, k}^{\sharp} \boldsymbol{E}_{1,2}, & 0<k<2 N \\ \boldsymbol{P}_{e, 2 N-1}^{\sharp} \boldsymbol{E}_{2,1}, & k=2 N .\end{cases}
$$

Then $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ is also a matrix CQF which satisfies Condition SA.
Proof. From the definition of $\boldsymbol{P}_{o, k}^{\sharp}$ in (2.1), we obtain

$$
\begin{equation*}
\boldsymbol{P}_{o}^{\sharp}(\omega)=\boldsymbol{P}_{e}^{\sharp}(\omega) \boldsymbol{M}(\omega), \tag{2.2}
\end{equation*}
$$

where

$$
\boldsymbol{M}(\omega)=\left[\begin{array}{cc}
0 & 1  \tag{2.3}\\
e^{-j \omega} & 0
\end{array}\right]
$$

is a unitary matrix. To show that $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ is a matrix CQF is equivalent to showing that $\boldsymbol{P}_{o}^{\sharp}(\omega)$ satisfies (1.3). We have

$$
\begin{aligned}
& \boldsymbol{P}_{o}^{\sharp}(\omega)\left(\boldsymbol{P}_{o}^{\sharp}(\omega)\right)^{*}+\boldsymbol{P}_{o}^{\sharp}(\omega+\pi)\left(\boldsymbol{P}_{o}^{\sharp}(\omega+\pi)\right)^{*} \\
& \quad=\boldsymbol{P}_{e}^{\sharp}(\omega) \boldsymbol{M}(\omega) \boldsymbol{M}^{*}(\omega)\left(\boldsymbol{P}_{e}^{\sharp}(\omega)\right)^{*}+\boldsymbol{P}_{e}^{\sharp}(\omega+\pi) \boldsymbol{M}(\omega+\pi) \boldsymbol{M}^{*}(\omega+\pi)\left(\boldsymbol{P}_{e}^{\sharp}(\omega+\pi)\right)^{*} \\
& \quad=\boldsymbol{P}_{e}^{\sharp}(\omega)\left(\boldsymbol{P}_{e}^{\sharp}(\omega)\right)^{*}+\boldsymbol{P}_{e}^{\sharp}(\omega+\pi)\left(\boldsymbol{P}_{e}^{\sharp}(\omega+\pi)\right)^{*} \\
& \quad=\boldsymbol{I}_{2 \times 2} .
\end{aligned}
$$

We note here that

$$
\begin{equation*}
\boldsymbol{A} \boldsymbol{M}(\omega)=e^{-j \omega} \boldsymbol{M}(-\omega) \boldsymbol{A} . \tag{2.4}
\end{equation*}
$$

Applying Lemma 1(iv) for $\boldsymbol{P}_{e}^{\sharp}(\omega)$, we have from (2.2) and (2.4)

$$
\begin{aligned}
\boldsymbol{P}_{o}^{\sharp}(\omega) & =\boldsymbol{A} \boldsymbol{P}_{e}^{\sharp}(-\omega) \boldsymbol{A} e^{-j(2 N-1) \omega} \boldsymbol{M}(\omega) \\
& =\boldsymbol{A} \boldsymbol{P}_{e}^{\sharp}(-\omega) e^{-j \omega} \boldsymbol{M}(-\omega) \boldsymbol{A} e^{j \omega} e^{-j 2 N \omega} \\
& =\boldsymbol{A} \boldsymbol{P}_{e}^{\sharp}(-\omega) \boldsymbol{M}(-\omega) \boldsymbol{A} e^{-j 2 N \omega} \\
& =\boldsymbol{A} \boldsymbol{P}_{o}^{\sharp}(-\omega) \boldsymbol{A} e^{-j 2 N \omega} .
\end{aligned}
$$

Therefore, by Lemma 1, $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ satisfies condition (1.11). Given that $\boldsymbol{P}_{e}(0)$ has the form (1.12), one can obtain $\boldsymbol{P}_{o}(0)=\boldsymbol{P}_{e}(0) \boldsymbol{S}=\operatorname{diag}(1,-\lambda)$, where $|\lambda|<1$. It is easy to see that (1.10) is satisfied. Thus $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ satisfies Condition SA.


FIG. 1. Relationship between even- and odd-length matrix $\operatorname{CQFs}\left\{\boldsymbol{P}_{e, k}^{\sharp}\right\}$ and $\left\{\boldsymbol{P}_{o, k}^{\sharp}\right\}$. Note here identical columns are aligned vertically and marked with arrows.

Remark 2.1. Denote $\Delta_{1}=\left(\boldsymbol{E}_{11}-\boldsymbol{E}_{12}+\boldsymbol{E}_{21}-\boldsymbol{E}_{22}\right) / 2, \Delta_{2}=\left(\boldsymbol{E}_{11}+\boldsymbol{E}_{12}-\boldsymbol{E}_{21}-\right.$ $\left.\boldsymbol{E}_{22}\right) / 2$. Under the assumption of Theorem 1, the matrix sequence $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ is given by

$$
\boldsymbol{P}_{o, k}= \begin{cases}\boldsymbol{P}_{e, 0} \Delta_{2}, & k=0,  \tag{2.5}\\ \boldsymbol{P}_{e, k-1} \Delta_{1}+\boldsymbol{P}_{e, k} \Delta_{2}, & 0<k<2 N, \\ \boldsymbol{P}_{e, 2 N-1} \Delta_{1}, & k=2 N\end{cases}
$$

THEOREM 2. Let $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ be an odd-length matrix CQF which satisfies Condition SA. Construct the matrix sequence $\left\{\boldsymbol{P}_{e, k}^{\sharp}\right\}_{k=0}^{2 N-1}$ from $\left\{\boldsymbol{P}_{o, k}^{\sharp}\right\}_{k=0}^{2 N}$ using the following rules:

$$
\begin{equation*}
\boldsymbol{P}_{e, k}^{\sharp}=\boldsymbol{P}_{o, k}^{\sharp} \boldsymbol{E}_{2,1}+\boldsymbol{P}_{o, k+1}^{\sharp} \boldsymbol{E}_{1,2}, \quad k=0,1, \ldots, 2 N-1 . \tag{2.6}
\end{equation*}
$$

Then $\left\{\boldsymbol{P}_{e, k}\right\}_{k=0}^{2 N-1}$ is also a matrix CQF which satisfies Condition SA.
Proof. From (2.6), we obtain $\boldsymbol{P}_{e}^{\sharp}(\omega)$ as

$$
\boldsymbol{P}_{e}^{\sharp}(\omega)=\boldsymbol{P}_{o}^{\sharp}(\omega) \boldsymbol{M}^{*}(\omega),
$$

where $\boldsymbol{M}(\omega)$ is defined in (2.3). The rest of the proof is similar to that for Theorem 1.
Remark 2.2. The matrix sequence $\boldsymbol{P}_{e, k}$ is given by

$$
\begin{equation*}
\boldsymbol{P}_{e, k}=\boldsymbol{P}_{o, k} \Delta_{1}+\boldsymbol{P}_{o, k+1} \Delta_{2}, \quad k=0,1, \ldots, 2 N-1 . \tag{2.7}
\end{equation*}
$$

Remark 2.3. The essence of Theorems 1 and 2 is captured in Fig. 1. As an illustration, suppose that we wish to obtain a length- $(2 N+1)$ matrix CQF from a length- $2 N$ matrix $\mathrm{CQF}\left\{\boldsymbol{P}_{e, k}\right\}_{k=0}^{2 N-1}$. First transform it to $\left\{\boldsymbol{P}_{e, k}^{\sharp}\right\}_{k=0}^{2 N-1}$ with elements $\boldsymbol{P}_{e, k}^{\sharp}=\left(p_{i j}^{\sharp}(k)\right)_{i, j=1}^{2}$. Then use these matrix elements to construct $\left\{\boldsymbol{P}_{o, k}^{\sharp}\right\}_{k=0}^{2 N}$ in the manner shown in Fig. 1. Finally, transform the resultant sequence to a length- $(2 N+1)$ matrix $\operatorname{CQF}\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$.

## 3. EXPLICIT CONSTRUCTION OF MULTIWAVELET FUNCTIONS

Once a lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}$ is obtained, the corresponding highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}$ needs to be constructed. For scalar wavelets, this problem has a simple solution: the highpass sequence is obtained by order reversing and sign alternating the lowpass sequence. For multiwavelets, Lawton et al. [14] have proposed a general approach for constructing the highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}$ using matrix extension techniques. However, for orthonormal multiwavelet systems satisfying Condition SA, explicit formulations are possible. We present in this section two explicit formulations for constructing the highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}$ directly in terms of $\left\{\boldsymbol{P}_{k}\right\}$. We will mainly consider even-length multiwavelet systems here as Theorem 1 can be used to generate odd-length multiwavelet systems.

Proposition 1. Let the lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ be a matrix CQF satisfying Condition SA. If $\boldsymbol{P}_{k} \boldsymbol{A} \boldsymbol{P}_{2 N-1-k-2 i}^{T}, k=0,1, \ldots, N-i-1, i=0,1, \ldots, N-1$, are symmetric matrices, then the highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ can be obtained from $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ as follows:

$$
\begin{equation*}
\boldsymbol{Q}_{k}=(-1)^{k+1} \boldsymbol{P}_{2 N-1-k} \boldsymbol{A}, \quad k=0,1, \ldots, 2 N-1 . \tag{3.1}
\end{equation*}
$$

Proof. We only need to prove that $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ and $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ satisfy the PR conditions (1.7) and (1.8). For $i \in \mathbb{Z}$,

$$
\begin{aligned}
\sum_{k=0}^{2 N-1-2 i} \boldsymbol{P}_{k} \boldsymbol{Q}_{k+2 i}^{T}= & \sum_{k=0}^{2 N-1-2 i} \boldsymbol{P}_{k}\left((-1)^{k+2 i+1} \boldsymbol{P}_{2 N-1-k-2 i} \boldsymbol{A}\right)^{T} \\
= & \left(\sum_{k=0}^{N-1-i}+\sum_{k=N-i}^{2 N-1-2 i}\right) \boldsymbol{P}_{k}(-1)^{k+2 i+1} \boldsymbol{A} \boldsymbol{P}_{2 N-1-k-2 i}^{T} \\
= & \sum_{k=0}^{N-1-i} \boldsymbol{P}_{k}(-1)^{k+1} \boldsymbol{A} \boldsymbol{P}_{2 N-1-k-2 i}^{T} \\
& +\sum_{k=0}^{N-1-i} \boldsymbol{P}_{2 N-1-2 i-k}(-1)^{k} \boldsymbol{A} \boldsymbol{P}_{k}^{T}
\end{aligned}
$$

Using the assumption that $\boldsymbol{P}_{k} \boldsymbol{A} \boldsymbol{P}_{2 N-1-2 i-k}^{T}$ are symmetric matrices for all $k=0,1, \ldots$, $N-i-1, i=0,1, \ldots, N-1$, we have

$$
\begin{equation*}
\sum_{k=0}^{2 N-1-2 i} \boldsymbol{P}_{k} \boldsymbol{Q}_{k+2 i}^{T}=0_{2 \times 2}, \quad i \in \mathbb{Z} \tag{3.2}
\end{equation*}
$$

It is easy to show that $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ given by (3.1) satisfies the PR condition (1.8).
Remark 3.1. It can be shown that $\boldsymbol{P}_{k} \boldsymbol{A} \boldsymbol{P}_{2 N-1-2 i-k}^{T}=\boldsymbol{P}_{2 N-1-2 i-k} \boldsymbol{A} \boldsymbol{P}_{k}^{T}$ is equivalent to $\boldsymbol{P}_{k}^{\sharp} \boldsymbol{S}\left(\boldsymbol{P}_{2 N-1-2 i-k}^{\sharp}\right)^{T}=\boldsymbol{P}_{2 N-1-2 i-k}^{\sharp} \boldsymbol{S}\left(\boldsymbol{P}_{k}^{\sharp}\right)^{T}$ for all $k=0,1, \ldots, N-i-1, i=$ $0,1, \ldots, N-1$. Applying Proposition 1, we have

$$
\boldsymbol{Q}_{k}^{\sharp}=(-1)^{k} \boldsymbol{P}_{2 N-1-k}^{\sharp} \boldsymbol{S}, \quad k=0,1, \ldots, 2 N-1 .
$$

Note that $\boldsymbol{A} \boldsymbol{S}=-\boldsymbol{S} \boldsymbol{A}$, i.e., $\boldsymbol{A} \boldsymbol{S}$ is an antisymmetric matrix. As a consequence of Proposition 1, symmetric-antisymmetric orthonormal multiscaling functions lead to symmetric-antisymmetric orthonormal multiwavelet functions. This is because for $k=$ $0,1, \ldots, 2 N-1$,

$$
\begin{aligned}
\boldsymbol{Q}_{k} & =(-1)^{k+1} \boldsymbol{P}_{2 N-1-k} \boldsymbol{A} \\
& =(-1)^{k+1} \boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{S} \boldsymbol{A} \\
& =(-1)^{k} \boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{A} \boldsymbol{S}=\boldsymbol{S} \boldsymbol{Q}_{2 N-1-k} \boldsymbol{S}
\end{aligned}
$$

We see that under the given condition of Proposition 1 the highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ can be obtained easily from the lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ in a manner similar to that for the scalar case. Here, if $\left\{\boldsymbol{P}_{0}, \boldsymbol{P}_{1}, \ldots, \boldsymbol{P}_{2 N-2}, \boldsymbol{P}_{2 N-1}\right\}$ is the lowpass sequence, then $\left\{-\boldsymbol{P}_{2 N-1} \boldsymbol{A}, \boldsymbol{P}_{2 N-2} \boldsymbol{A}, \ldots,-\boldsymbol{P}_{1} \boldsymbol{A}, \boldsymbol{P}_{0} \boldsymbol{A}\right\}$ is the corresponding highpass sequence.

As an example, the length-4 symmetric-antisymmetric orthonormal multiwavelet systems constructed by Chui and Lian [1] and Jiang [12] satisfy the condition of Proposition 1; thus their highpass sequences can in fact be obtained directly from the lowpass sequences via (3.1). It should be pointed out, however, that not all lowpass sequences will satisfy the condition given in Proposition 1. Consider the following example:

Example 3.1. Let

$$
\boldsymbol{P}_{0}=\left[\begin{array}{cc}
\frac{4-\sqrt{15}}{8} & \frac{1}{8} \\
\frac{-4+\sqrt{15}}{8} & \frac{-1}{8}
\end{array}\right], \quad \boldsymbol{P}_{1}=\left[\begin{array}{cc}
\frac{4+\sqrt{15}}{8} & \frac{1}{8} \\
\frac{4+\sqrt{15}}{8} & \frac{1}{8}
\end{array}\right]
$$

$\boldsymbol{P}_{2}=\boldsymbol{S} \boldsymbol{P}_{1} \boldsymbol{S}$, and $\boldsymbol{P}_{3}=\boldsymbol{S} \boldsymbol{P}_{0} \boldsymbol{S}$. This length-4 lowpass sequence generates an orthonormal MRA with symmetric-antisymmetric multiscaling functions. However, the highpass sequence cannot be obtained directly via Proposition 1.

In the following, we will present another method for constructing highpass sequences directly from lowpass sequences satisfying another condition.

Proposition 2. Let the lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ be a matrix CQF satisfying Condition SA. If the sum of antidiagonal elements is zero for each of the matrices $\boldsymbol{P}_{2 k} \boldsymbol{A} \boldsymbol{P}_{2 k+2 i+1}^{T}-\boldsymbol{P}_{2 k+1} \boldsymbol{A} \boldsymbol{P}_{2 k+2 i}^{T}, k=0,1, \ldots, N-i-1, i=0,1, \ldots, N-1$, then the highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ can be obtained from $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ as follows:

$$
\begin{equation*}
\boldsymbol{Q}_{2 k}=-\boldsymbol{P}_{2 k+1} \boldsymbol{A}, \quad \boldsymbol{Q}_{2 k+1}=\boldsymbol{P}_{2 k} \boldsymbol{A}, \quad k=0,1, \ldots, N-1 . \tag{3.3}
\end{equation*}
$$

Proof. We only need to prove that $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ and $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ satisfy the PR conditions (1.7)-(1.8). Denote

$$
\boldsymbol{B}_{k, i}:=\boldsymbol{P}_{2 k} \boldsymbol{A} \boldsymbol{P}_{2 k+2 i+1}^{T}-\boldsymbol{P}_{2 k+1} \boldsymbol{A} \boldsymbol{P}_{2 k+2 i}^{T}
$$

$k=0,1, \ldots, N-i-1, i=0,1, \ldots, N-1$. Applying $\boldsymbol{P}_{k}=\boldsymbol{S} \boldsymbol{P}_{2 N-1-k} \boldsymbol{S}$, we have

$$
\begin{equation*}
\boldsymbol{B}_{k, i}=-\boldsymbol{S} \boldsymbol{B}_{N-k-i-1, i}^{T} \boldsymbol{S} \tag{3.4}
\end{equation*}
$$

This leads to

$$
\begin{aligned}
\sum_{k=0}^{2 N-1-2 i} \boldsymbol{P}_{k} \boldsymbol{Q}_{k+2 i}^{T} & =-\sum_{k=0}^{N-1-i} \boldsymbol{B}_{k, i} \\
& = \begin{cases}-\sum_{k=0}^{\frac{N-i}{2}-1}\left(\boldsymbol{B}_{k, i}+\boldsymbol{B}_{N-k-i-1, i}\right), & N-i \text { is even } \\
-\boldsymbol{B}_{\frac{N-i-1}{2}, i}-\sum_{k=0}^{\frac{N-i-1}{2-1}-1}\left(\boldsymbol{B}_{k, i}+\boldsymbol{B}_{N-k-i-1, i}\right), & N-i \text { is odd. }\end{cases}
\end{aligned}
$$

Given that the sum of antidiagonal elements of $\boldsymbol{B}_{k, i}$ is zero, we can show from (3.4) that

$$
\boldsymbol{B}_{k, i}+\boldsymbol{B}_{N-k-i-1, i}=\boldsymbol{B}_{k, i}-\boldsymbol{S} \boldsymbol{B}_{k, i}^{T} \boldsymbol{S}=\mathbf{0}_{2 \times 2},
$$

$k=0,1, \ldots, N-i-1, i=0,1, \ldots, N-1$. Furthermore, for the case $N-i$ is odd, (3.4) holds for $k=(N-i-1) / 2$. Letting $n=(N-i-1) / 2$, we have

$$
\boldsymbol{B}_{n, i}+\boldsymbol{B}_{n, i}=\boldsymbol{B}_{n, i}-\boldsymbol{S} \boldsymbol{B}_{n, i}^{T} \boldsymbol{S}=\mathbf{0}_{2 \times 2}
$$

Therefore $\boldsymbol{B}_{(N-1-i) / 2, i}=\mathbf{0}_{2 \times 2}$.
Hence, $\sum_{k=0}^{2 N-1-2 i} \boldsymbol{P}_{k} \boldsymbol{Q}_{k+2 i}^{T}=\mathbf{0}_{2 \times 2}$ for both even and odd $N-i$. For PR condition (1.8), one needs only apply (3.3) to show that it is satisfied.

Remark 3.2. The lowpass sequence in Example 3.1 satisfies the condition of Proposition 2, and thus the corresponding highpass sequence can be obtained via (3.3).

Remark 3.3. The assumption of Proposition 2 implies that all diagonal elements of the matrices $\boldsymbol{P}_{2 k}^{\sharp} \boldsymbol{S}\left(\boldsymbol{P}_{2 k+2 j+1}^{\sharp}\right)^{T}-\boldsymbol{P}_{2 k+1}^{\sharp} \boldsymbol{S}\left(\boldsymbol{P}_{2 k+2 j}^{\sharp}\right)^{T}$ are equal and

$$
\boldsymbol{Q}_{2 k}^{\sharp}=\boldsymbol{P}_{2 k+1}^{\sharp} \boldsymbol{S}, \quad \boldsymbol{Q}_{2 k+1}^{\sharp}=-\boldsymbol{P}_{2 k}^{\sharp} \boldsymbol{S}, \quad k=0,1, \ldots, N-1 .
$$

As in the case of Proposition 1, the highpass matrix sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ constructed in Proposition 2 also satisfies the relation $\boldsymbol{Q}_{k}=\boldsymbol{S} \boldsymbol{Q}_{2 N-1-k} S$. Hence the corresponding multiwavelet functions also form a symmetric-antisymmetric pair.

In the previous section, we presented a relationship between even- and odd-length matrix CQFs. The same relationship exists between the corresponding highpass matrix filters of these matrix CQFs, and we can apply it to generate odd-length highpass matrix filters from even-length highpass matrix filters.

Proposition 3. Let $\left\{\boldsymbol{P}_{e, k}\right\}_{k=0}^{2 N-1}$ be an even-length matrix CQF satisfying Condition SA, and $\left\{\boldsymbol{Q}_{e, k}\right\}_{k=0}^{2 N-1}$ be the corresponding highpass matrix filter satisfying (1.10) and (1.11). Then one can construct an odd-length matrix CQF $\left\{\boldsymbol{P}_{o, k}\right\}_{k=0}^{2 N}$ satisfying Condition SA using (2.5). The corresponding odd-length highpass matrix filter $\left\{\boldsymbol{Q}_{o, k}\right\}_{k=0}^{2 N}$ can be obtained from $\left\{\boldsymbol{Q}_{e, k}\right\}_{k=0}^{2 N-1}$ using (2.5) (with symbol $\boldsymbol{P}$ replaced by $\boldsymbol{Q}$ throughout).

In the following, we will make use of Propositions 1, 2, and 3 to give explicit formulations of highpass sequences $\left\{\boldsymbol{Q}_{k}\right\}$ for symmetric-antisymmetric orthonormal multiwavelet systems with support lengths of 2,3 , and 4 .

EXAmple 3.2. Let $\{\phi, \psi\}$ be a length-2 orthonormal multiwavelet system with its lowpass matrix sequence satisfying Condition SA. Then this lowpass sequence is given by

$$
\boldsymbol{P}_{0}=\left[\begin{array}{cc}
1 & 0 \\
\cos \theta & \sin \theta
\end{array}\right], \quad \boldsymbol{P}_{1}=\left[\begin{array}{cc}
1 & 0 \\
-\cos \theta & \sin \theta
\end{array}\right] .
$$

Here $\boldsymbol{P}_{0}$ and $\boldsymbol{P}_{0}$ satisfy the conditions of both Propositions 1 and 2 . Applying either Proposition 1 or 2, we obtain the corresponding highpass matrices

$$
\boldsymbol{Q}_{0}=\left[\begin{array}{cc}
0 & -1 \\
-\sin \theta & \cos \theta
\end{array}\right], \quad \boldsymbol{Q}_{1}=\left[\begin{array}{cc}
0 & 1 \\
\sin \theta & \cos \theta
\end{array}\right]
$$

where $\theta \in[0,2 \pi) \backslash\left\{\frac{\pi}{2}, \frac{3 \pi}{2}\right\}$ for the transition operator to satisfy Condition E .
Example 3.3. We can apply Proposition 3 to the above family of length- 2 multiwavelet systems to obtain length-3 symmetric-antisymmetric orthonormal multiwavelet systems with lowpass and highpass matrix sequences $\left\{\boldsymbol{P}_{0}, \boldsymbol{P}_{1}, \boldsymbol{P}_{2}\right\}$ and $\left\{\boldsymbol{Q}_{0}, \boldsymbol{Q}_{1}, \boldsymbol{Q}_{2}\right\}$ given by

$$
\begin{array}{ll}
\boldsymbol{P}_{0}=\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} \sin \gamma & \frac{\sqrt{2}}{2} \sin \gamma
\end{array}\right], & \boldsymbol{P}_{1}=\left[\begin{array}{cc}
1 & 0 \\
0 & -\sqrt{2} \cos \gamma
\end{array}\right] \\
\boldsymbol{P}_{2}=\left[\begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{\sqrt{2}}{2} \sin \gamma & \frac{\sqrt{2}}{2} \sin \gamma
\end{array}\right], & \boldsymbol{Q}_{0}=\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
-\frac{\sqrt{2}}{2} \cos \gamma & -\frac{\sqrt{2}}{2} \cos \gamma
\end{array}\right] \\
\boldsymbol{Q}_{1}=\left[\begin{array}{cc}
-1 & 0 \\
0 & \sqrt{2} \sin \gamma
\end{array}\right], & \boldsymbol{Q}_{2}=\left[\begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
\frac{\sqrt{2}}{2} \cos \gamma & -\frac{\sqrt{2}}{2} \cos \gamma
\end{array}\right],
\end{array}
$$

where $\gamma=\frac{\pi}{4}-\theta, \theta \in[0,2 \pi) \backslash\left\{\frac{\pi}{2}, \frac{3 \pi}{2}\right\}$.
When the parameter $\gamma=2 \pi-\arcsin (\sqrt{14} / 4)$, we obtain the length 3 symmetricantisymmetric orthonormal multiwavelet system constructed by Chui and Lian in [1].

Example 3.4. For length-4 orthonormal symmetric-antisymmetric multiwavelet systems, a possible lowpass matrix sequence is given by

$$
\begin{aligned}
\boldsymbol{P}_{0} & =\frac{\beta}{1+\beta^{2}}\left[\begin{array}{cc}
\beta & -1 \\
\beta\left(\beta^{2} \tau^{2}-2 \tau-1\right) / \lambda & -\left(\beta^{2} \tau^{2}+2 \beta^{2} \tau-1\right) / \lambda
\end{array}\right] \\
\boldsymbol{P}_{1} & =\frac{1}{1+\beta^{2}}\left[\begin{array}{cc}
1 & -\beta \\
\left(\beta^{2} \tau^{2}+2 \beta^{2} \tau-1\right) / \lambda & -\beta\left(\beta^{2} \tau^{2}-2 \tau-1\right) / \lambda
\end{array}\right],
\end{aligned}
$$

$\boldsymbol{P}_{2}=\boldsymbol{S} \boldsymbol{P}_{1} \boldsymbol{S}, \boldsymbol{P}_{3}=\boldsymbol{S} \boldsymbol{P}_{0} \boldsymbol{S}$ and $\lambda=1+\beta^{2} \tau^{2}$. Here $\boldsymbol{P}_{k}, k=0,1,2,3$, satisfy the condition of Proposition 1. Hence we obtain the corresponding highpass matrix sequence:

$$
\begin{aligned}
& \boldsymbol{Q}_{0}=\frac{\beta}{1+\beta^{2}}\left[\begin{array}{cc}
-1 & -\beta \\
\left(\beta^{2} \tau^{2}+2 \beta^{2} \tau-1\right) / \lambda & \beta^{2}\left(\beta^{2} \tau^{2}-2 \tau-1\right) / \lambda
\end{array}\right] \\
& \boldsymbol{Q}_{1}=\frac{1}{1+\beta^{2}}\left[\begin{array}{cc}
\beta & 1 \\
-\beta\left(\beta^{2} \tau^{2}-2 \tau-1\right) / \lambda & -\left(\beta^{2} \tau^{2}+2 \beta^{2} \tau-1\right) / \lambda
\end{array}\right]
\end{aligned}
$$

$\boldsymbol{Q}_{2}=\boldsymbol{S} \boldsymbol{Q}_{1} \boldsymbol{S}$ and $\boldsymbol{Q}_{3}=\boldsymbol{S} \boldsymbol{Q}_{0} \boldsymbol{S}$.

When $\beta=-(10-3 \sqrt{10}) /(5 \sqrt{6}-2 \sqrt{15})$ and $\tau=(45-9 \sqrt{15}) /(15-6 \sqrt{10}-10 \sqrt{6}+$ $5 \sqrt{15}$ ), we obtain the length-4 symmetric-antisymmetric orthonormal multiwavelet system constructed in [1].

## 4. ORTHONORMAL MULTIWAVELETS AND RELATED SCALAR WAVELETS

The aim of this section is to expound the relationship between symmetric-antisymmetric orthonormal multiwavelet systems and scalar wavelets. We will show that scalar orthonormal wavelets can be used to generate symmetric-antisymmetric orthonormal multiwavelet systems with multiplicity 2 . The following discussion will focus only on constructing evenlength multiwavelet systems as odd-length multiwavelet systems can be obtained through Proposition 3.

A scalar sequence $\left\{a_{k}\right\}_{k \in \mathbb{Z}}$ is referred to as a scalar $C Q F$ if it satisfies

$$
\begin{equation*}
\sum_{k} a_{k} a_{k+2 i}=2 \delta_{i, 0}, \quad i \in \mathbb{Z} \tag{4.2}
\end{equation*}
$$

We will only be dealing with scalar CQFs from orthonormal scalar wavelets and as such these scalar CQFs will have at least one vanishing moment meaning that any such scalar CQF $\left\{a_{k}\right\}_{k \in \mathbb{Z}}$ satisfies

$$
\begin{equation*}
\sum_{k}(-1)^{k} a_{k}=0 \tag{4.3}
\end{equation*}
$$

For the rest of this section, all scalar CQFs referred to will be assumed to have at least one vanishing moment.

For any orthonormal symmetric-antisymmetric multiwavelet system satisfying Condition SA, the refinement masks $\boldsymbol{P}(\omega)$ satisfies (1.12). This means that $\boldsymbol{P}^{\sharp}(\omega)$ at $\omega=0$ have the form

$$
\boldsymbol{P}^{\sharp}(0)=\frac{1}{2}\left[\begin{array}{ll}
1+\lambda & 1-\lambda \\
1-\lambda & 1+\lambda
\end{array}\right],
$$

with $|\lambda|<1$. The value of $\lambda$ plays a very important role in the design of new multiwavelet systems for signal processing, especially for image compression (see [21, 24]). The special case $\lambda=0$ can arise as shown in the following.

Consider the lowpass matrix sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ of any orthonormal multiwavelet system satisfying Condition SA. By Lemma 1(ii), the transformed lowpass matrix sequence $\left\{\boldsymbol{P}_{k}^{\sharp}\right\}_{k=0}^{2 N-1}$ has the form

$$
\boldsymbol{P}_{k}^{\sharp}=\left[\begin{array}{cc}
a_{2 k} & a_{2 k+1}  \tag{4.4}\\
a_{4 N-2 k-1} & a_{4 N-2 k-2}
\end{array}\right], \quad k=0,1, \ldots, 2 N-1 .
$$

That is, if one forms a scalar sequence $\left\{a_{k}\right\}_{k=0}^{4 N-1}$ using the elements from the first rows of the matrices $\boldsymbol{P}_{k}^{\sharp}, k=0,1, \ldots, 2 N-1$, then the second rows are obtained by reversing this sequence. If this scalar sequence is a scalar CQF , then

$$
\sum_{k=0}^{2 N-1} a_{2 k}=1, \quad \sum_{k=0}^{2 N-1} a_{2 k+1}=1
$$

and this implies that

$$
\boldsymbol{P}^{\sharp}(0)=\left[\begin{array}{ll}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array}\right],
$$

and thus $\lambda=0$. That is, $\boldsymbol{P}(0)$ or $\boldsymbol{P}^{\sharp}(0)$ is singular, when the related scalar sequence $\left\{a_{k}\right\}$ is the lowpass sequence of an orthonormal scalar wavelet. This special case corresponds to the good multifilter properties for discrete multiwavelet transforms discussed in [21] which also demonstrated the importance of such properties for image compression purposes.

Assuming that $\left\{\boldsymbol{P}_{k}^{\sharp}\right\}_{k=0}^{2 N-1}$ is a matrix CQF, the following conditions on $\left\{a_{k}\right\}_{k=0}^{4 N-1}$ can be derived:

$$
\begin{array}{ll}
\sum_{k=0}^{4 N-1-4 i} a_{k} a_{k+4 i}=2 \delta_{i, 0}, & i \in \mathbb{Z} \\
\sum_{k=0}^{4 N-1-4 i} a_{k} a_{4 N-1-4 i-k}=0, & i \in \mathbb{Z} \\
\sum_{k=4 i}^{4 N-1} a_{k} a_{4 N-1+4 i-k}=0, & i \in \mathbb{Z} . \tag{4.7}
\end{array}
$$

Clearly if a scalar sequence satisfies Eqs. (4.5)-(4.7), then the corresponding matrix CQF can be constructed via (4.4). Note that a scalar CQF automatically satisfies (4.5), but not the other two.

We will examine two possible methods of obtaining such sequences from the lowpass sequences of scalar orthonormal wavelets.

### 4.1. Construction of Orthonormal Multiwavelets from Scalar Wavelets

Method 1. In the first method, we take a scalar CQF $\left\{h_{k}\right\}_{k=0}^{2 N-1}$ and double its length by inserting pairs of zeros to give $\left\{h_{0}, h_{1}, 0,0, \ldots, h_{2 N-2}, h_{2 N-1}, 0,0\right\}$ or $\left\{0,0, h_{0}, h_{1}, 0\right.$, $\left.0, \ldots, h_{2 N-2}, h_{2 N-1}\right\}$. Clearly, either one of the new sequences satisfies Eqs. (4.5)-(4.7). The following lemma contains this result.

LEMMA 4. Let $\left\{h_{k}\right\}_{k=0}^{2 N-1}$ be a scalar CQF. Construct a length- $4 N$ sequence $\left\{a_{k}\right\}_{k=0}^{4 N-1}$ as follows:

$$
\begin{align*}
& a_{4 k}=\frac{1}{2}(1-\xi) h_{2 k}, \quad a_{4 k+1}=\frac{1}{2}(1-\xi) h_{2 k+1}, \quad a_{4 k+2}=\frac{1}{2}(1+\xi) h_{2 k}, \\
& a_{4 k+3}=\frac{1}{2}(1+\xi) h_{2 k+1}, \quad \xi= \pm 1, k=0,1, \ldots, N-1 \tag{4.8}
\end{align*}
$$

Then this length $-4 N$ sequence is a scalar CQF.
The proof only involves verifying (4.2) and (4.3). This length- $4 N$ sequence can be used to construct a length- $2 N$ orthonormal multiwavelet system as shown in the following theorem.

THEOREM 3. Let $\left\{a_{k}\right\}_{k=1}^{4 N-1}$ be a scalar CQF obtained from (4.8). Use it to construct a matrix sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N}-1$ via (4.4). Then $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ is a matrix CQF and the corresponding highpass matrix sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ can be constructed via Proposition 2 . Furthermore, if the transition operator $\boldsymbol{T}_{\boldsymbol{P}}$ satisfies Condition $E$, then $\left\{\boldsymbol{P}_{k}, \boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ generates a length- $2 N$ orthonormal multiwavelet system satisfying Condition $S A$.

Proof. As mentioned earlier, the scalar CQF $\left\{a_{k}\right\}_{k=0}^{4 N-1}$ satisfies Eqs. (4.5)-(4.7), and this implies that $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ is a matrix CQF. On the other hand, all diagonal elements of $\boldsymbol{P}_{2 k}^{\sharp} \boldsymbol{S}\left(\boldsymbol{P}_{2 N-1-k-2 j}^{\sharp}\right)^{T}-\boldsymbol{P}_{2 k+1}^{\sharp} \boldsymbol{S}\left(\boldsymbol{P}_{2 N-k-2 j}^{\sharp}\right)^{T}$ are zeros; thus we can apply Proposition 2 to obtain the highpass matrix sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$.

Method 2. In the following, we give another approach to generate scalar sequences satisfying Eqs. (4.5)-(4.7).

Consider a scalar sequence $\left\{b_{k}\right\}_{k=0}^{4 N-1}$ where

$$
\begin{equation*}
b_{2 k+1}=\tau(-1)^{k+1} b_{2 k}, \quad \tau= \pm 1, k=0,1, \ldots, 2 N-1 . \tag{4.9}
\end{equation*}
$$

Such a sequence satisfies Eqs. (4.6) and (4.7) right away. Now if this scalar sequence is a CQF, then Eq. (4.5) is also satisfied. It can be easily verified that the lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}$ constructed via (4.4) satisfies the condition of Proposition 1, and thus the corresponding highpass sequence can be obtained via (3.1). We express the second method in the following theorem.

THEOREM 4. Let $\left\{b_{k}\right\}_{k=0}^{4 N-1}$ be a scalar CQF satisfying Eq. (4.9). Use it to construct a matrix sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ via (4.4). Then $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ is a matrix CQF and the corresponding highpass matrix sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ can be constructed via Proposition 1 . Furthermore, if the transition operator $\boldsymbol{T}_{\boldsymbol{P}}$ satisfies Condition $E$, then $\left\{\boldsymbol{P}_{k}, \boldsymbol{Q}_{k}\right\}_{k=0}^{2 N-1}$ generates a length-2N orthonormal multiwavelet system satisfying Condition SA.

Proof. The proof is similar to that for Theorem 3. We note that $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ in this case satisfies the condition given in Proposition 1.

How does one generate a length- $4 N$ scalar CQF satisfying (4.9)? One can construct this CQF from scratch using methods such as spectral factorization or lattice factorization. But a simple and direct method, which exploits any existing length- $2 N$ scalar CQFs, can be found in the following lemma.

Lemma 5. Let $\left\{h_{k}\right\}_{k=0}^{2 N-1}$ be a length- $2 N$ scalar CQF. Construct a length $-4 N$ sequence $\left\{b_{k}\right\}_{k=0}^{4 N-1}$ satisfying $b_{2 k+1}=\tau(-1)^{k+1} b_{2 k}, \tau= \pm 1, k=0,1, \ldots, 2 N-1$, as follows:

$$
\begin{align*}
b_{4 k} & =\frac{1}{2}\left(h_{2 k}-\tau h_{2 k+1}\right), \quad b_{4 k+2}=\frac{1}{2}\left(h_{2 N-2-2 k}+\tau h_{2 N-1-2 k}\right), \\
k & =0,1, \ldots, N-1 . \tag{4.10}
\end{align*}
$$

Then this length- $4 N$ sequence is a scalar CQF. Conversely, let $\left\{b_{k}\right\}_{k=0}^{4 N-1}$ be a length- $4 N$ scalar CQF satisfying (4.9). Construct a length-2N sequence $\left\{h_{k}\right\}_{k=0}^{2 N=1}$ as follows:

$$
\begin{equation*}
h_{2 k}=b_{4 N-2-4 k}+b_{4 k}, \quad h_{2 k+1}=\tau\left(b_{4 N-2-4 k}-b_{4 k}\right), \quad k=0,1, \ldots, N-1 . \tag{4.11}
\end{equation*}
$$

Then the sequence $\left\{h_{k}\right\}_{k=0}^{2 N-1}$ is a scalar CQF.
We will omit the proof of this lemma as it involves only verifying (4.2) and (4.3).
Given a length- $2 N$ scalar CQF, one can use either method 1 or 2 to construct a length$2 N$ matrix CQF. In fact, the relationship between the scalar and matrix CQFs given in the above lemma is simple enough for us to directly give the lowpass matrix sequences $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ in terms of the lowpass scalar sequence $\left\{b_{k}\right\}_{k=0}^{4 N-1}$ for both methods 1 and 2. By Lemma 5 and (1.14), we obtain for method 2,

$$
\begin{align*}
\boldsymbol{P}_{2 k}= & \frac{1}{2}\left[\begin{array}{cc}
(1-\tau) b_{4 k}+(1+\tau) b_{4 N-4 k-2} & -(1+\tau) b_{4 k}+(1-\tau) b_{4 N-4 k-2} \\
-(1-\tau) b_{4 k}+(1+\tau) b_{4 N-4 k-2} & (1+\tau) b_{4 k}+(1-\tau) b_{4 N-4 k-2}
\end{array}\right] \\
\boldsymbol{P}_{2 k+1}= & \frac{1}{2}\left[\begin{array}{cc}
(1+\tau) b_{4 k+2}+(1-\tau) b_{4 N-4 k-4} & -(1-\tau) b_{4 k+2}+(1+\tau) b_{4 N-4 k-4} \\
-(1+\tau) b_{4 k+2}+(1-\tau) b_{4 N-4 k-4} & (1-\tau) b_{4 k+2}+(1+\tau) b_{4 N-4 k-4}
\end{array}\right] \\
& \tau= \pm 1 . \tag{4.12}
\end{align*}
$$

Note that the scalar sequence $\left\{b_{k}\right\}$ should be computed using the same $\tau$ as appeared together in (4.12).

For method 1, (4.8) and (4.4) are first used to obtain $\boldsymbol{P}_{k}$ 's in terms of $\left\{h_{k}\right\}$. Then, in order to illustrate a link with method 2, Eqs. (4.11) are used to express these $\boldsymbol{P}_{k}$ 's in terms of $\left\{b_{k}\right\}_{k=0}^{4 N-1}$. When $\xi=-1$, we have

$$
\begin{aligned}
\boldsymbol{P}_{2 k}= & \frac{1}{2}\left[\begin{array}{cc}
(1-\tau) b_{4 k}+(1+\tau) b_{4 N-4 k-2} & -(1+\tau) b_{4 k}-(1-\tau) b_{4 N-4 k-2} \\
-(1-\tau) b_{4 k}-(1+\tau) b_{4 N-4 k-2} & (1+\tau) b_{4 k}+(1-\tau) b_{4 N-4 k-2}
\end{array}\right] \\
\boldsymbol{P}_{2 k+1}= & \frac{1}{2}\left[\begin{array}{cc}
(1+\tau) b_{4 k+2}+(1-\tau) b_{4 N-4 k-4} & (1-\tau) b_{4 k+2}+(1+\tau) b_{4 N-4 k-4} \\
(1+\tau) b_{4 k+2}+(1-\tau) b_{4 N-4 k-4} & (1-\tau) b_{4 k+2}+(1+\tau) b_{4 N-4 k-4}
\end{array}\right] \\
& \tau= \pm 1 .
\end{aligned}
$$

The matrix CQFs for $\xi=1$ can be obtained from those constructed with $\xi=-1$ by interchanging $\boldsymbol{P}_{2 k}$ with $\boldsymbol{P}_{2 k+1}, k=0,1, \ldots, N-1$.

Clearly for both methods, the two elements in each column of every $\boldsymbol{P}_{k}$ have the same magnitude. It can be shown that this is also true for the corresponding highpass sequences constructed using Propositions 1 and 2. Furthermore, if one has constructed a lowpass sequence using method 2 for $\tau=-1$, then a lowpass sequence for method 1 with $\xi=-1$ can be obtained easily by flipping the sign of the $(1,2)$ elements of all $\boldsymbol{P}_{k}$ 's for method 2. For the case when $\tau=1$, one only needs to flip the sign of the $(2,1)$ elements.

Remark 4.2. For method 2, the matrix CQFs obtained with the two possible values of $\tau$ are related: if $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{2 N-1}$ is obtained with one value of $\tau$, then $\left\{\boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{S}\right\}_{k=0}^{2 N-1}$ is obtained with the other value of $\tau$.

Remark 4.3. For method 1, every coefficient matrix of the matrix CQFs obtained is singular.

For symmetric-antisymmetric orthonormal multiwavelet systems considered in this section, there are two associated approximation order: one is for the multiscaling functions of the multiwavelet system (MAP), and another is for the corresponding scaling function of the related (length- $4 N$ ) scalar wavelet (SAP).

Remark 4.4. If the matrix $\operatorname{CQF}\left\{\boldsymbol{P}_{k}\right\}$ has a MAP of $m$ and a SAP of $n$, then the matrix CQFs $\left\{\boldsymbol{S} \boldsymbol{P}_{k}\right\},\left\{\boldsymbol{P}_{k} \boldsymbol{S}\right\}$, and $\left\{\boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{S}\right\}$ have their MAP and SAP at least equal to 1 . It can be easily shown that $\left\{\boldsymbol{S} \boldsymbol{P}_{k}\right\}$ always has the same SAP as $\left\{\boldsymbol{P}_{k}\right\}$, and that $\left\{\boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{S}\right\}$ always has the same MAP as $\left\{\boldsymbol{P}_{k}\right\}$. The MAP for $\left\{\boldsymbol{S} \boldsymbol{P}_{k}\right\}$, the SAP for $\left\{\boldsymbol{S} \boldsymbol{P}_{k} \boldsymbol{S}\right\}$, as well as the MAP and SAP for $\left\{\boldsymbol{P}_{k} \boldsymbol{S}\right\}$ need not be the same as those for $\left\{\boldsymbol{P}_{k}\right\}$, except when these MAPs/SAPs are equal to 1 .

### 4.2. Length-4 Multiwavelets Derived from Scalar Wavelets

Here we illustrate the process of deriving length-4 multiwavelet systems from length-4 scalar orthonormal wavelets. Daubechies [5] gave an example of an one-parameter length-4 orthonormal scalar wavelet with the lowpass sequence $\left\{h_{k}\right\}_{k=0}^{3}$ given by

$$
h_{0}=\frac{v(v-1)}{v^{2}+1}, \quad h_{1}=\frac{1-v}{v^{2}+1}, \quad h_{2}=\frac{1+v}{v^{2}+1}, \quad h_{3}=\frac{v(v+1)}{v^{2}+1} .
$$

This sequence is a scalar CQF, and the parameter $v$ can be used for various filter design purposes.

We consider the derivation using method 2 first. Applying (4.10) to $\left\{h_{k}\right\}_{k=0}^{3}$ results in the length- 8 scalar CQF $\left\{b_{k}\right\}_{k=0}^{7}$ where

$$
\begin{array}{lll}
b_{0}=\frac{(v-1)(v+\tau)}{2\left(1+v^{2}\right)}, & b_{1}=\frac{-\tau(v-1)(\nu+\tau)}{2\left(1+v^{2}\right)}, & b_{2}=\frac{(1+\nu)(1+\nu \tau)}{2\left(1+v^{2}\right)} \\
b_{3}=\frac{\tau(1+\nu)(1+\nu \tau)}{2\left(1+v^{2}\right)}, & b_{4}=\frac{(1+v)(1-v \tau)}{2\left(1+v^{2}\right)}, & b_{5}=\frac{-\tau(1+\nu)(1-v \tau)}{2\left(1+v^{2}\right)} \\
b_{6}=\frac{(v-1)(v-\tau)}{2\left(1+v^{2}\right)}, & b_{7}=\frac{\tau(v-1)(v-\tau)}{2\left(1+v^{2}\right)} . &
\end{array}
$$

We then apply (4.12) to obtain the corresponding length-4 matrix lowpass sequence $\left\{\boldsymbol{P}_{k}\right\}_{k=0}^{3}$ where

$$
\begin{align*}
& \boldsymbol{P}_{0}=\frac{1}{2\left(1+v^{2}\right)}\left[\begin{array}{cc}
(v-1)^{2} & \tau\left(1-v^{2}\right) \\
\tau(v-1)^{2} & v^{2}-1
\end{array}\right] \\
& \boldsymbol{P}_{1}=\frac{1}{2\left(1+v^{2}\right)}\left[\begin{array}{cc}
(v+1)^{2} & \tau\left(1-v^{2}\right) \\
-\tau(v+1)^{2} & 1-v^{2}
\end{array}\right]  \tag{4.13}\\
& \boldsymbol{P}_{2}=\boldsymbol{S} \boldsymbol{P}_{1} \boldsymbol{S} \quad \text { and } \quad \boldsymbol{P}_{3}=\boldsymbol{S} \boldsymbol{P}_{0} \boldsymbol{S} .
\end{align*}
$$

From the above length-4 matrix CQF for method 2, one obtains the matrix CQFs for method 1 through flipping signs of appropriate matrix elements and other CQFs by applying Lemma 2 . Thus one can construct a family of length-4 orthonormal multiwavelet systems (SA4) through varying the value of the parameter $v$ for different filter design goals [20].

Let us consider a few specific members of this family. The matrix CQF given in (4.13) has both MAP and SAP equal to 1 . The parameter $v$ can be used to increase either approximation order to two. To increase MAP to two, for both $\tau= \pm 1$, we require $\nu=(-2 \pm \sqrt{19}) / 3$. To increase SAP to two, we require $\nu= \pm \sqrt{15} / 5$ for $\tau=1$, and $\nu= \pm \sqrt{15} / 3$ for $\tau=-1$.

Figure 2 b shows the multiscaling functions of SA4 member (SA4a) constructed under method 2 with $\mathrm{SAP}=2$ and $\mathrm{MAP}=1(\tau=1$ and $v=\sqrt{15} / 5)$. The length -4 and length -8 scalar scaling functions are also shown in Fig. 2a.

One can generate another SA4 member (SA4b) by applying case (i) of Lemma 2 to SA4a. Figures 2 c and 2d depict SA4b. This member has $\mathrm{SAP}=2$ and $\mathrm{MAP}=1$. The corresponding length -8 scalar scaling function for SA4b is actually a left-right flip of that for SA4a.

Applying case (iii) of Lemma 2 gives a third member (SA4c) which appears in Figs. 3e and 3 f. As expected, here the multiscaling function $\phi_{1}$ is the same as that for SA4a,


FIGURE 2

(a)

(b)

(c)

FIGURE 3
whereas $\phi_{2}$ is a left-right flip of that for SA4a. However, the SAP for SA4c is 1 and not 2 as for SA4a, due to the fact that the corresponding length-8 scalar wavelets are no longer the same as that for SA4a. The transition operators for SA4a, SA4b, and SA4c have been verified to satisfy Condition E via numerical techniques.

It is interesting to note the changes in the regularities of the scaling functions involved as the construction proceed from length- 4 to length- 8 scalar scaling functions and to the length-4 multiscaling functions. Using estimates for Sobolev regularity [11, 23], in the case of SA4a, the regularities change from 0.7075 for length-4 scalar, to 1.5094 for length-8 scalar, and finally to 0.9919 for SA4a itself. For SA4b, the changes are from 0.3390 to 1.5094 and finally to 0.2485 . For SA4c, the changes are from 0.7075 to 0.2360 and finally to 0.9919 .

For any of the above length-4 orthonormal multiwavelet systems, one can apply Proposition 3 to obtain a length- 5 orthonormal multiwavelet system.

### 4.3. Even-Length Multiwavelets Derived from Scalar Wavelets

In this section, we will consider the construction of length- $2 N$ multiwavelet systems from related length- $2 N$ scalar wavelets. Parameterized representations of length $-2 N$ scalar CQFs are first given using lattice factorization. Examples of multiwavelet systems are then given for $N=3$.

Let $H_{0}(z)$ and $H_{1}(z)$ be the $z$-transform of the lowpass and highpass filters associated with an orthonormal scalar wavelet. The polyphase components of $H_{0}(z)$ will be denoted by $H_{00}(z)$ and $H_{01}(z)$, that is,

$$
H_{0}(z)=H_{00}\left(z^{2}\right)+z^{-1} H_{01}\left(z^{2}\right)=\frac{1}{2} \sum_{k=0}^{2 N-1} h_{k} z^{-k}
$$

The polyphase components give the even and odd indexed coefficients of $H_{0}(z)$ separately. Similarly, the polyphase components of $H_{1}(z)$ will be denoted by $H_{10}(z)$ and $H_{11}(z)$ so that

$$
H_{1}(z)=H_{10}\left(z^{2}\right)+z^{-1} H_{11}\left(z^{2}\right)=\frac{1}{2} \sum_{k=0}^{2 N-1} g_{k} z^{-k}
$$

where $g_{k}=(-1)^{k} h_{2 N-1-k}, k=0,1, \ldots, 2 N-1$. The polyphase matrix of the scalar wavelet filter is then defined as

$$
\boldsymbol{H}_{p}^{N}(z)=\left[\begin{array}{ll}
H_{00}(z) & H_{10}(z)  \tag{4.14}\\
H_{01}(z) & H_{11}(z)
\end{array}\right] .
$$

An important result by Vaidyanathan [22] is that the polyphase matrix $\boldsymbol{H}_{p}^{N}(z)$ can be written as

$$
\begin{equation*}
\boldsymbol{H}_{p}^{N}(z)=\frac{1}{\sqrt{2}} \boldsymbol{R}_{0} \prod_{j=1}^{N-1} \boldsymbol{D}(z) \boldsymbol{R}_{j} \tag{4.15}
\end{equation*}
$$

where

$$
\boldsymbol{D}(z)=\left[\begin{array}{cc}
1 & 0 \\
0 & z^{-1}
\end{array}\right], \quad \text { and } \quad \boldsymbol{R}_{j}=\left[\begin{array}{cc}
\cos \alpha_{j} & -\sin \alpha_{j} \\
\sin \alpha_{j} & \cos \alpha_{j}
\end{array}\right], \quad j=0,1, \ldots, N-1
$$

This lattice parameterization gives $\boldsymbol{H}_{p}^{N}(z)$ as a function of angles. Note that $\boldsymbol{H}_{p}^{N}(z)$ is of degree $N-1$. Therefore the filters $H_{0}(z)$ and $H_{1}(z)$ are of degree $2 N-1$. For the filter to be orthonormal and have at least one vanishing moment, it is necessary that $H_{0}(1)=1$ and $H_{0}(-1)=0$. Equivalently, these conditions can be expressed in terms of the angles $\alpha_{j}, j=0,1, \ldots, N-1$, such that

$$
\begin{equation*}
\sum_{j=0}^{N-1} a_{j}=2 n \pi+\frac{\pi}{4}, \quad n \in \mathbb{Z} \tag{4.16}
\end{equation*}
$$

It is clear that we still have $N-1$ degrees of freedom in the remaining $\alpha$ 's. From this ( $N-1$ )-parameter scalar sequence, we then construct length- $2 N$ matrix CQFs using methods 1 and 2.

In the following presentation, we find it convenient to replace the parameters $\alpha_{i}$ with $\beta_{i}=\tan \alpha_{i}, i=0,1, \ldots, N-1$, when expressing matrix filter coefficients associated with the multiwavelet systems.

SA6 multiwavelet systems. For $N=3$, taking $\alpha_{0}=2 n \pi+\frac{\pi}{4}-\alpha_{1}-\alpha_{2}, \beta_{1}=\tan \alpha_{1}$, $\beta_{2}=\tan \alpha_{2}$, and $\gamma=\left(1+\beta_{1}^{2}\right)\left(1+\beta_{2}^{2}\right)$ results in the length- 6 scalar CQF with coefficients:

$$
\begin{array}{ll}
h_{0}=\left(\beta_{1}+\beta_{2}-\beta_{1} \beta_{2}+1\right) / \gamma, & h_{1}=-\left(\beta_{1}+\beta_{2}+\beta_{1} \beta_{2}-1\right) / \gamma \\
h_{2}=\beta_{1}\left(\beta_{1}-1\right) /\left(1+\beta_{1}^{2}\right), & h_{3}=\beta_{1}\left(\beta_{1}+1\right) /\left(1+\beta_{1}^{2}\right) \\
h_{4}=\beta_{2}\left(\beta_{1}+\beta_{2}+\beta_{1} \beta_{2}-1\right) / \gamma, & h_{5}=\beta_{2}\left(\beta_{1}+\beta_{2}-\beta_{1} \beta_{2}+1\right) / \gamma .
\end{array}
$$

Applying Lemma 5, we obtain the length- 12 scalar CQF with coefficients:

$$
\begin{aligned}
b_{0} & =\left(\left(\beta_{1}+\beta_{2}\right)(\tau+1)+\left(\beta_{1} \beta_{2}-1\right)(\tau-1)\right) /(2 \gamma) \\
b_{2} & =\beta_{2}\left(\left(\beta_{1}+\beta_{2}\right)(\tau+1)-\left(\beta_{1} \beta_{2}-1\right)(\tau-1)\right) /(2 \gamma) \\
b_{4} & =\beta_{1}\left(\beta_{1}-1-\tau\left(\beta_{1}+1\right)\right) /\left(2\left(1+\beta_{1}^{2}\right)\right) \\
b_{6} & =\beta_{1}\left(\beta_{1}-1+\tau\left(\beta_{1}+1\right)\right) /\left(2\left(1+\beta_{1}^{2}\right)\right) \\
b_{8} & =\beta_{2}\left(\left(\beta_{1}+\beta_{2}\right)(-\tau+1)+\left(\beta_{1} \beta_{2}-1\right)(\tau+1)\right) /(2 \gamma) \\
b_{10} & =\left(\left(\beta_{1}+\beta_{2}\right)(-\tau+1)-\left(\beta_{1} \beta_{2}-1\right)(\tau+1)\right) /(2 \gamma),
\end{aligned}
$$

and $b_{2 k+1}=\tau(-1)^{k+1} b_{2 k}, k=0,1, \ldots, 5$. From this length-12 scalar sequence $\left\{b_{k}\right\}_{k=0}^{11}$, methods 1 and 2, and Lemma 2 are then used to construct the SA6 family of symmetric-antisymmetric orthonormal multiwavelet systems. For members constructed under method 2 with $\tau=-1$, the coefficient matrices are

$$
\begin{aligned}
& \boldsymbol{P}_{0}=\frac{1}{\left(\beta_{1}^{2}+1\right)\left(\beta_{2}^{2}+1\right)}\left[\begin{array}{cc}
-\beta_{1} \beta_{2}+1 & -\tau\left(\beta_{1}+\beta_{2}\right) \\
-\tau\left(\beta_{1} \beta_{2}-1\right) & \beta_{1}+\beta_{2}
\end{array}\right] \\
& \boldsymbol{P}_{1}=\frac{\beta_{2}}{\left(\beta_{1}^{2}+1\right)\left(\beta_{2}^{2}+1\right)}\left[\begin{array}{cc}
\beta_{1}+\beta_{2} & -\tau\left(-\beta_{1} \beta_{2}+1\right) \\
-\tau\left(\beta_{1}+\beta_{2}\right) & \beta_{1} \beta_{2}-1
\end{array}\right]
\end{aligned}
$$

TABLE 1
For Three Members of SA6 Families of Symmetric-Antisymmetric Orthonormal Multiwavelets, the Required Conditions on the Parameters $\beta_{1}$ and $\beta_{2}$ for Achieving Various Combinations of Multiwavelet Approximation Order (MAP) and Scalar Wavelet Approximation Order (SAP) are Shown.

|  | $(\mathrm{SAP}, \mathrm{MAP})$ | Condition on parameters |
| :--- | :---: | :---: |
| SA6a | $(1,3)$ | $17 \beta_{1}^{4}+256 \beta_{1}^{3}+1298 \beta_{1}^{2}+2816 \beta_{1}+705=0$, |
|  | $\beta_{2}=\frac{-1}{9984}\left(1088 \beta_{1}^{3}+9856 \beta_{1}^{2}+40256 \beta_{1}+41600\right)$. |  |
| SA6b | $(2,2)$ | $\beta_{2}^{2}-16 \beta_{2}+1=0$, |
|  | $\beta_{1}=8-\beta_{2}$. |  |
| SA6c | $(3,1)$ | $\beta_{2}^{4}-32 \beta_{2}^{3}+258 \beta_{2}^{2}-544 \beta_{2}+129=0$, |
|  | $\beta_{1}=\frac{-1}{34}\left(\beta_{2}^{3}-28 \beta_{2}^{2}+163 \beta_{2}-28\right)$. |  |

$$
\boldsymbol{P}_{2}=\frac{\beta_{1}}{\beta_{1}^{2}+1}\left[\begin{array}{cc}
\beta_{1} & \tau \\
\beta_{1} \tau & -1
\end{array}\right]
$$

$\boldsymbol{P}_{k}=\boldsymbol{S} \boldsymbol{P}_{5-k} \boldsymbol{S}, k=3,4,5$. The highpass sequence $\left\{\boldsymbol{Q}_{k}\right\}_{k=0}^{5}$ is given by (3.1). This branch of SA6 family, with two parameters yet to be determined, has both MAP and SAP at least equal to 1 . Table 1 shows the conditions imposed on $\beta_{1}$ and $\beta_{2}$ for members of this branch to achieve three possible combinations of approximation orders. Up to four possible pairs of values of $\beta_{1}$ and $\beta_{2}$ are available for each combination of approximation orders. Among the possible pairs, the one which yields multiscaling functions with highest regularity is selected for each multiwavelet system. Figure 3 shows the graphs of the multiscaling functions so obtained for three members of this branch of SA6. The transition operators of all associated refinement masks have been verified to satisfy Condition E through direct computation. Although the graphs of these three sets of multiscaling functions look alike, their estimates of the Sobolev regularity differ: 1.5034, 1.4986, and 0.9995 for SA6a, SA6b, and SA6c, respectively. For any of the above length-6 orthonormal multiwavelet systems, one can apply Proposition 3 to obtain a length- 7 orthonormal multiwavelet system.

## 5. CONCLUSIONS

In this paper, we studied a class of orthonormal multiwavelet systems with multiplicity 2 consisting of pairs of symmetric-antisymmetric multiscaling and multiwavelet functions. We first showed how a length- $2 N$ multiwavelet system can be constructed from a length- $(2 N+1)$ multiwavelet system and vice versa. Next, we presented two explicit formulations for the direct construction of multiwavelet functions from their corresponding scaling functions. Examples of families of parameterized orthonormal multiwavelet systems and their relationships with some other multiwavelet systems reported in previous expositions were also given. We then investigated the relationship between a symmetric-antisymmetric orthonormal multiwavelet system and its related orthonormal scalar wavelet. There, we presented two methods for the construction of symmetricantisymmetric orthonormal multiwavelet systems from a given orthonormal scalar wavelet.

For each of the methods, we also showed how length- $2 N$ multiwavelet systems can be derived from related length- $2 N$ scalar wavelets. Examples were given for length-4 and length-6 multiwavelet systems. It is noteworthy that the SA4a multiwavelet system was shown in [21] to perform better in image compression than the GHM multiwavelet, the Chui and Lian's length-4 multiwavelet, and both length-4 and length-8 Daubechies scalar wavelets.

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